PATRIZIA LONGOBARDI[†] **and MERCEDE MAJ**[‡]

[†]Dipartimento di Matematica e Applicazioni "R. Caccioppoli"
via Cintia, Monte S. Angelo — 80126 Napoli — Italy
longobar@matna2.dma.unina.it

[‡]Dipartimento di Matematica e Informatica
via Salvator Allende — 84081 Baronissi (Salerno) — Italy
maj@matna2.dma.unina.it

The authors are members of C.N.R.–G.N.S.A.G.A., Italy

ON SOME CLASSES OF ORDERABLE GROUPS*

Conferenza tenuta il giorno 9 Novembre 1998

1 Introduction

A group G is said to be orderable (an O-group) if there exists a full order relation $\leq$ on the set G such that $a \leq b$ implies $xay \leq xby$, for all $a, b, x, y \in G$, i.e. the order on G is compatible with the product of G . For example an infinite cyclic group is an O-group.

A subgroup H of an O-group G can be ordered by taking the restriction to H of any order on G and the cartesian product $C = Cr_{\alpha \in \Omega} G_\alpha$ of the orderable groups G_α can be ordered with the lexicographic order. Hence the class of O-groups is $\{S, C, D, R\}$-closed.

*1991 Mathematics Subjects Classification: Primary: 20D15, Secondary: 20D60.
This work has been partially supported by M.U.R.S.T.

Furthermore this class is L-closed (i.e. a locally O-group is an O-group) by a result of B. H. Neumann (see [27]) and w-closed (i.e. the restricted wreath product of two O-groups is an O-group), as B. H. Neumann and Fuchs and Sasiada proved in [26] and in [11].

Therefore a torsion-free abelian group is an O-group, and it is possible to show that a torsion-free nilpotent group is an O-group (see [2], theorem 2.2.4).

In a quite different direction A. Vinogradov proved in [34] that a free product of two O-groups is an O-group.

If G is a group with a full order $\leq$ compatible with the product, we will say that $(G, \leq)$ is an ordered group; we will denote the identity element by e and, if $a \in G$, we will say that a is positive if $a > e$, negative if $a < e$.

If a is positive, then we have $a^n > e$, for any $n \in N$, thus an O-group is always torsion-free. Therefore the class of O-groups is not closed under quotients. Furthermore this class is not closed under extensions, for instance the semidirect product $\langle a \rangle \rtimes \langle b \rangle$ of two infinite cyclic groups, with $a^b = a^{-1}$, can never be an O-group, since it contains some element other than e that is conjugate to its inverse. For the same reason, the unrestricted wreath product of two non-trivial O-groups can never be an O-group.

A subgroup C of an ordered group $(G, \leq)$ is said to be convex under $\leq$ if $x \in C$, whenever $e \leq x \leq c$, for some $c \in C$.

Obviously $\{e\}$ and G are convex subgroups of G, and, if C is a convex subgroup, then every conjugate C^g of C is convex. It is clear from the definition that all convex subgroups of an ordered group form, by inclusion, a totally ordered set, closed under intersection and union.

A relatively convex subgroup of an O-group G is, by definition, a subgroup convex under some order on G.

Relatively convex subgroups play important roles, for example the quotient G/N of an O-group G is an O-group if and only if N is a normal relatively convex subgroup of G.

Orders on a group G in which $\{e\}$ and G are the only convex subgroups are very well-known. They are exactly the archimedean orders, where an order on G is called an archimedean order if and only if, for any $a, b \in G$, $a > e$, $b > e$, there exists a positive integer

n such that $b < a^n$. By a result of O. Hölder (see [15]) an order on G is an archimedean order if and only if G is order-isomorphic to a subgroup of the additive group of the real numbers under the natural order.

If C and D are convex subgroups of an ordered group G, $C < D$, and there is not a convex subgroup H of G such that $C < H < D$, we call $C \to D$ a convex jump in G.

It is easy to prove that, if $C \to D$ is a convex jump in an ordered group G, then $C \lhd D$ and D/C is order-isomorphic to a subgroup of the additive group of the real numbers.

A group G is called right-orderable (or an RO-group) if there exists a full order relation on the set G such that $a \leq b$ implies $ac \leq bc$ for all a, b, c in G. As before we use the term "right ordered group" to denote a group with an order that satisfies the previous property.

Obviously an O-group is an RO-group and an abelian RO-group is an O-group. The class of RO-groups is closed under operations S, L, D, C, R, W. But this class is also closed under extensions (see [2], theorem 7.3.2).

For instance the semidirect product $\langle a \rangle \rtimes \langle b \rangle$ of two infinite cyclic groups with $a^b = a^{-1}$ is an RO-group which is not an O-group.

If $(G, \leq)$ is a right ordered group we define a subgroup C of G to be convex if, for every $g \in G$, $c \in C$, $e \leq g \leq c$ implies $g \in C$. If $C < D$ are convex subgroups, we say that $C \to D$ is a convex jump if there is no proper convex subgroup of G between C and D.

As with ordered groups, the set of all convex subgroups of a right ordered group $(G, \leq)$ forms a chain, but it is not anymore true that $C \lhd D$ and D/C is abelian, if $C \to D$ is a convex jump (see [20] for an example).

A right order $\leq$ is called a Conrad order if $C \lhd D$ and D/C is archimedean, for any convex jump $C \to D$. A group G is called a Conrad group if there exists a Conrad order on G. Obviously an O-group is a Conrad group; Conrad orders have been studied by P. Conrad in [9]. In this paper he proved that a right order is a Conrad order if and only if for each pair of positive elements a, b there exists a positive integer n such that $a^n b > a$.

It is not difficult to show that if G is a Conrad group, then every finitely generated non-trivial subgroup of G has an infinite cyclic factor group, i.e. G is locally indicable. Conversely Burns and Hale proved in [5] that in a locally indicable group it is possible to define a right order which is a Conrad order.

It has been an open question for some time whether the class of right ordered groups and the class of locally indicable groups coincide. Then G. Bergman in [4] and V. M. Tararin in [32] constructed right ordered groups that are not locally indicable.

However for some classes of groups the two concepts are equivalent: in [7], generalizing a previous result by A. H. Rhemtulla, I. M. Chiswell and P. Kropholler proved that a finite extension of a solvable group is right orderable if and only if it is locally indicable. This result has been extended to the class of periodic extensions of radical groups by V.M. Tararin in [32].

Ordered and right ordered groups have been considered by many authors (see for instance [2, 4, 10, 16, 19, 20]).

Here we look at some particular classes of ordered and right ordered groups. We survey some recent results about ordered and right ordered groups satisfying a non-trivial semigroup law, we show that an O-group with this property is nilpotent, while an RO-group in this class is locally indicable. Furthemore we survey results about ordered and right ordered n-Engel groups. We show that an n-Engel O-group is nilpotent (see [16]), and that an RO 4-Engel group satisfies a non-trivial semigroup law and then it is nilpotent. We point out that it is still an open question whether an RO n-Engel group is nilpotent. Finally we recall some open questions about ordered and right ordered groups satisfying some finiteness conditions.

2 Orderable and right orderable groups satisfying a non-trivial semigroup law

Let $F = F(x_1, x_2, ..., x_n)$ be the free group on the letters x_1, x_2,..., x_n.

A word $u = u(x_1, x_2, ..., x_n)$ is called a positive word if it does not involve x_i^{-1}, for any $i \in \{1, 2, ..., n\}$.

If $u = u(x_1, x_2, ..., x_n)$, $v = v(x_1, x_2, ..., x_n) \in F$, a law $u = v$ is said to be a semigroup law (or a positive law) if u, v are positive words.

Obviously every group of finite exponent satisfies a non-trivial positive law, and every abelian group satisfies the positive law $xy = yx$.

B.H. Neumann and T. Taylor in [28] and A.I. Mal'cev in [25] proved that a nilpotent group of class c satisfies the positive law

$$u_c(x,y) = v_c(x,y)\,,$$

where the words $u_i(x,y)$, $v_i(x,y)$ are recursively defined by

$$\begin{aligned} u_0(x,y) &= x \\ v_0(x,y) &= y \\ u_{i+1}(x,y) &= u_i(x,y)v_i(x,y) \\ v_{i+1}(x,y) &= v_i(x,y)u_i(x,y)\,. \end{aligned}$$

It follows that a group which is a finite exponent extension of a nilpotent group satisfies a non-trivial semigroup law.

It is easy to prove that if a group G satisfies a non-trivial semigroup law, then G satisfies a non-trivial semigroup law in two letters. Hence groups that satisfy a non-trivial semigroup law belong to the class of groups which contain no free semigroups in two generators. We will denote the latter class by $\mathfrak{C}$. Thus a group $G \in \mathfrak{C}$ if, and only if, for every pair (a,b) of elements of G, there is a relation of the form

$$a^{r_1}b^{s_1}\ldots.a^{r_j}b^{s_j} = b^{m_1}a^{n_1}\ldots.b^{m_k}a^{n_k}\,, \tag{2.1}$$

where r_i, s_i, m_i, n_i are all non-negative and r_1, m_1 are positive integers. We call $j+k$ the width of the relation (2.1) and the sum $r_1 + \ldots + r_j + n_1 + \ldots + n_k$ the exponent of a in the relation.

Groups with a non-trivial semigroup law and, more generally, groups in the class $\mathfrak{C}$ have been studied by many authors. By the result of Mal'cev's, every group which is a periodic extension of a locally nilpotent group is in the class $\mathfrak{C}$. In [30] J. Rosenblatt proved that, conversely, a finitely generated soluble group in $\mathfrak{C}$ is nilpotent-by-finite. The same result for finitely generated soluble groups that satisfy a non-trivial semigroup law had been proved in [21] by J.A. Lewin and T. Lewin. A. Shalev proved in [31] that every finitely generated residually finite group with a non-trivial semigroup law is nilpotent-by-finite. More generally, Y. Kim and A. H. Rhemtulla proved in [17] that if G is a locally graded group in $\mathfrak{C}$ and there is a bound n such that for all ordered pairs of elements of G there is a relation (2.1) with $\exp a + \exp b = n$, then G is (locally) nilpotent-by-finite. We point out that in general a group in $\mathfrak{C}$ is not locally soluble-by-finite, as has been proved by A. Yu. Ol'shanski and A. Storozev in [29]. More results about groups satisfying a positive law have been recently obtained by R. Burns, O. Macedonska and Y. Medvedev in [6].

We will show that O-groups satisfying a semigroup law behave like soluble groups, and that RO-groups in $\mathfrak{C}$ are locally indicable. First we remark that

LEMMA 2.1 *Let $(G, \leq)$ be an ordered group, $a, b \in G$, $n \in \mathbb{Z}$. Then*

i. *from $[a^n, b] = 1$, it follows $[a, b] = 1$,*

ii. *if G is nilpotent-by-finite, then G is nilpotent.*

PROOF.

i. Assume, for example $b^{-1}ab > a$. Then $(b^{-1}ab)^2 > a(b^{-1}ab) > a^2$, and, by induction, $(b^{-1}ab)^s > a^s$, for every positive integer s, a contradiction.

ii. Let N be a nilpotent normal subgroup of finite index, say t, in G and write $\zeta_i(G)$ ($\zeta_i(N)$) the i-centre of G (of N). We show that $\zeta_k(N) = \zeta_k(G)$, by induction on k . Assume $\zeta_k(N) = \zeta_k(G)$; then for any $g \in G$, $a \in \zeta_{k+1}(N)$, we have $[a, g^t]\zeta_k(G) = \zeta_k(G)$. But $G/\zeta_k(G)$ is an ordered group (see [2] , Theorem 2.2.4), hence $[a, g] \in \zeta_k(G)$, by (i), and $a \in \zeta_{k+1}(G)$. Therefore $G/\zeta_c(G)$ is finite, for some integer c, and $G = \zeta_c(G)$, since $G/\zeta_c(G)$ is torsion-free. □

Now we will show that a finitely generated O-group satisfying a semigroup law is nilpotent. More generally, we show (see [22])

THEOREM 2.1 *Let G be an O-group. Suppose that there is a bound n such that for all pairs (a, b) of elements of G there is a relation (2.1) whose width is at most n. Then G is nilpotent of class bounded by a function of n.*

In order to prove Theorem 2.1, we need the following key Lemma

LEMMA 2.2 *If G is a group with no free non-abelian subsemigroups, then, for all a, b in G, the subgroup $\langle a\rangle^{\langle b\rangle}$ is finitely generated.*

PROOF. See [22], Lemma 1. □

Groups G in which $\langle a\rangle^{\langle b\rangle}$ is finitely generated for any a, b in G are called constrained, and have been studied by Y. K. Kim e A. Rhemtulla in [18], where they proved the following useful result

LEMMA 2.3 *Let G be a finitely generated restrained group. If $H \lhd G$ and G/H is cyclic, then H is finitely generated. In particular if G is soluble, then G is polycyclic.*

From Lemma 2.3 it follows easily the following

LEMMA 2.4 *Let C be a convex subgroup of an ordered group $(G, \leq)$. If $G \in \mathfrak{C}$, then C is normal in G.*

PROOF. Suppose not. Then either $C < C^x$ or $C^x < C$, for some $x \in G$, since the convex subgroups form a chain. Assume, for example, $C < C^x$.

Then there exists $a^x \in C^x \backslash C$, where $a \in C$.

By Lemma 2.2, $\langle a\rangle^{\langle x\rangle} = \langle a, a^{\pm x}, \dots, a^{\pm x^{n-1}}\rangle$, for some n, thus $\langle a\rangle^{\langle x\rangle} \subseteq C^{x^{n-1}}$, $a^{x^n} \in C^{x^{n-1}}$, and $a^x \in C$, a contradiction. □

Now we are able to prove Theorem 2.1.

PROOF OF THEOREM 2.1. Let $(G, \leq)$ be an ordered group. We may assume G finitely generated. Then every non-trivial finitely generated convex subgroup of G contains a proper maximal convex subgroup.

Put $K := \{C \lhd G | C$ convex, G/C nilpotent$\}$. Then G/K is a residually (torsion-free nilpotent) group. By theorem 3 of [22], G/K is nilpotent of bounded class. If $K = 1$, we have the result. Assume $K \neq 1$. Then K is finitely generated, by Lemma 2.3. Thus there exists a maximal convex subgroup $D \neq 1$, $D < K$. Then $D \lhd G$, by Lemma 2.4. Moreover K/D is a jump in the set of all convex subgroups of G, hence K/D is an abelian torsion-free group. Then G/D is soluble and G/D is a nilpotent torsion-free group, a contradiction since $D < K$. □

The answer to the following question is still unknown:

PROBLEM 1 *Is a finitely generated O-group with no free non-abelian subsemigroup nilpotent?*

Arguing as in the proof of Theorem 2.1, we can reduce the problem to residually (torsion-free nilpotent) groups. Thus Problem 1 reduces to

PROBLEM 2 *Let G be a finitely generated torsion-free group with no free non-abelian subsemigroup. Is G nilpotent?*

The structure of an RO-group can be quite complicated, even if G has no free non-abelian subsemigroup.

EXAMPLE. Let p be a prime, F be the free group of rank two and F/R be isomorphic to the Gupta-Sidki p-group constructed in [14]. Then F/R is an infinite residually finite p-group. Then F/R' is a residually (torsion-free solvable) group, and a residually (finite p-group). For all pairs (a,b) of elements in G, there is a relation (2.1) with $j = k = 1$, but G is not nilpotent-by-finite.

We have the following

THEOREM 2.2 *Let G be an RO-group. If G has no free non abelian subsemigroup, then G is locally indicable.*

PROOF. Let $\leq$ be a right order in G, and let a, b be positive elements in G. We show that $a^{\ n}b > a$, for some positive integer n, thus $\leq$ is a Conrad order and the result follows. If $b > a$, then $a^{\ n}b > b > a$, for all positive integers n, so assume $a > b$. By hypothesis

$$a^{r_1}b^{s_1}...a^{r_j}b^{s_j} = b^{m_1}a^{n_1}...b^{m_k}a^{n_k}\,, \qquad (2.1)$$

where r_i, s_i, m_i, n_i are all non-negative and s_j, n_k are positive.

If $a > a^m b$, for all $m = 0$, then $a^{r_1}b^{s_1} = (a^{r_1}b)b^{s_1-1} < ab^{s_1-1} < ab < a$. Continuing in this way, we get $a^{r_1}b^{s_1}...a^{r_j}b^{s_j} < a$. On the other side, $b^{m_1}a^{n_1}...b^{m_k} \geq e$, so that $b^{m_1}a^{n_1}...b^{m_k}a^{n_k} \geq a$, and we get a contradiction. □

Theorem 2.2, togheter with Theorem 3 of [22], gives the following

THEOREM 2.3 *If G is an RO-group and there is a bound n such that for all ordered pairs of elements of G there is a relation (2.1) with* $\exp(a) \leq n$, *then G is locally nilpotent-by-finite.*

The following recent result (see [24]) generalizes Theorem 2.2.

THEOREM 2.4 *Let G be an RO-group and assume that there exists an ascending series of normal subgroups of G with each factor with no free non-abelian subsemigroup. Then G is locally indicable.*

3 Orderable and right orderable k-Engel groups.

Let G be a group, $a, b \in G$, and $n \geq 0$ be an integer. We define the commutator $[a,_n b]$ by induction, by putting

$$[a,_0 b] = a\,, \qquad [a,_{i+1} b] = [[a,_i b], b]\,.$$

G is said to be a k-Engel group if $[a,_k b] = 1$, for all a, b in G.

Obviously 1-Engel groups are the abelian ones, and every nilpotent group of class c is a c-Engel group. Conversely, every finite k-Engel group is nilpotent, by a result of Zorn (see [37]). There exist infinite 3-Engel groups with trivial centre, but it is still an open question whether a k-Engel group is locally nilpotent. K. Gruenberg proved in [12] that a soluble k-Engel group is locally nilpotent and in [13] a similar result for linear k-Engel groups. J. Wilson in [34] proved that a residually finite k-Engel group is locally nilpotent, and Y. Kim and A. Rhemtulla extended this result to locally graded groups in [17].

Ordered k-Engel groups have been studied by Y. Kim and A. Rhemtulla in [16]. They proved the following

THEOREM 3.1 *A k-Engel O-group is nilpotent.*

We give a proof of Theorem 3.1 very similar to the proof of Theorem 2.1.

First of all we point out that

LEMMA 3.1 *Let G be a k-Engel group. Then the subgroup $\langle a\rangle^{\langle b\rangle}$ can be generated with k elements, for any a, b in G.*

PROOF. From $[a,_k b] = 1$, we get easily $\langle a^{\langle b\rangle}\rangle = \langle a, a^b, \dots, a^{b^{k-1}}\rangle$. □

From Lemma 3.1, arguing as in the proof of Lemma 2.4 we get

LEMMA 3.2 *Let C be a convex subgroup of an ordered group $(G, \leq)$. If G is a k-Engel group, then C is normal in G.*

Now we are able to prove Theorem 3.1.

PROOF OF THEOREM 3.1. Let $(G, \leq)$ be an ordered group. By a result of Zel'manov ([37]) a torsion-free nilpotent $k-$Engel group has nilpotency class bounded by a function of k. In particular a locally nilpotent torsion-free k-Engel group is nilpotent. Hence we can assume G finitely generated.

Let $K := \{C \lhd G | C$ convex, G/C nilpotent$\}$. Then G/K is a residually (torsion-free nilpotent) group, then by Zel'manov's result, G/K is nilpotent of bounded class. If $K = 1$, we have the result. Assume $K \neq 1$. Then K is finitely generated, by Lemma 2.3. Thus there exists a maximal convex subgroup $D \neq 1$, $D < K$. Then $D \lhd G$, by Lemma 3.2. Moreover $K \to D$ is a jump in the set of all convex subgroups of G, hence K/D is an abelian torsion-free group. Then G/D is soluble and G/D is a nilpotent torsion-free group, a contradiction since $D < K$. □

It is still an open question whether a right ordered k-Engel group is nilpotent. We have the following partial result (see [23])

THEOREM 3.2 *A 4-Engel right ordered group is nilpotent.*

PROOF. Let G be a 4-Engel group. Then $\langle a, a^b\rangle$ is nilpotent of class ≤ 4, for any a, b in G, by a result of M. Vaughan-Lee and G.Traustason (see [34]). Then $u_4(a, a^b) = v_4(a, a^b)$, where $u_4(x, y)$ and $v_4(x, y)$ are the Mal'cev's words defined before. Then, for any g, h in G , we have $u_4(gh, hg) = v_4(gh, hg)$, since gh and hg are conjugate. Then G satisfies a non-trivial semigroup identity and G is nilpotent by Theorem 2.2. □

Arguing as in the proof of Theorem 3.2 it is possible to show that a k-Engel group is nilpotent if it satisfies a non-trivial semigroup identity.

Therefore the following

PROBLEM 3 *Is any right ordered k-Engel group nilpotent?*

reduces to

PROBLEM 4 *Does every right ordered k-Engel group satisfy a non-trivial semigroup identity?*

It is also unknown the answer to the following

PROBLEM 5 *Is any right ordered k-Engel group locally indicable?*

Lemmas 2.2 and 3.2 are crucial in the proofs of Theorems 2.1 and 3.1. Therefore it is natural to ask the following questions.

PROBLEM 6 *What is possible to say about constrained ordered and right ordered groups?*

More generally we may ask

PROBLEM 7 *Is any ordered group with the maximal condition on subgroup solvable?*

References

[1] G. M. BERGMAN, *Right-orderable groups that are not locally indicable,* Pac. J. Math. **147** (1991) 243–248.

[2] R. BOTTO MURA AND A.H. RHEMTULLA, *Orderable Groups*, Marcel Dekker, New York–Basel 1977.

[3] R. BOTTO MURA AND A.H. RHEMTULLA, *Ordered solvable groups satisfying the maximal condition on isolated subgroups and groups with finitely many relatively convex subgroups*, J. Algebra **36** (1975) 38–45.

[4] R.T. BOTTO MURA, *Right-orderable policlyclic groups*, Canad. Math. Bull. **17** (1974) 175–178.

[5] R.G. Burns and V.W. Hale, *A note on group rings of certain torsion- free groups*, Can. Math. Bull. **15** (1972) 441–445.

[6] R.G. Burns, O. Macedonska, Y. Medvedev, *Groups satisfying semigroup laws, and nilpotent-by-Burnside varieties*, J. Algebra **195** (1997) 510–525.

[7] I.M. Chiswell and P.H. Kropholler, *Soluble right orderable groups are locally indicable*, Can. Math. Bull. **36** (1993) 22–29.

[8] P.M. Cohn, *Groups of order automorphisms of ordered sets*, Mathematika **4** (1957) 41–50.

[9] P.F. Conrad, *Right-ordered groups*, Mich. Math. J. **6** (1959) 267–275.

[10] L. Fuchs, *Partially Ordered Algebraic Systems*, Pergamon Press, New York–Oxford, 1963.

[11] L. Fuchs and E. Sasiada, *Note on orderable groups*, Ann. Univ. Sci. Budapest Eötvös Sect. Math. **7**, (1964) 13–17.

[12] K. Gruenberg, *The Engel elements of a soluble group*, Illinois J. Math. **3** (1959) 151–168.

[13] K. Gruenberg, *The Engel structure of linear groups,* J. Algebra **3** (1966) 291–303.

[14] N. Gupta and S. Sidki, *Some infinite p-groups*, Algebra i Logica **22** (1983) 584–589.

[15] O. Hölder, *Die Axiome der Quantität and die Lehre vom Mass.*, Ber. Verh. Sächs. Ges. Wiss. Leipzig, Math. Phys. Cl. **53** (1901) 1–64.

[16] Y. Kim and A.H. Rhemtulla, *Groups with Ordered Structures*, Proc. of “Groups Korea ’94”, de Gruyter, Berlin (1995) 199–210.

[17] Y. Kim and A.H. Rhemtulla, *On locally Graded Groups*, Proc. of “Groups Korea ’94”, de Gruyter, Berlin (1995) 189–198.

[18] Y.K. KIM AND A.H. RHEMTULLA, *Weak maximality condition and polycyclic groups*, Proc. Amer. Math. Soc. **123** (1995) 711–714.

[19] A.I. KOKORIN, V.M. KOPYTOV, *Fully Ordered Groups*, John Wiley & Sons, New York–Toronto, 1974.

[20] V.M. KOPITOV AND N. YA. MEDVEDEV, *Right Ordered Groups*, Siberian School of Algebra and Logic, Plenum Publishing Corporation, New York–London–Moscow, 1996.

[21] J.A. LEWIN AND T. LEWIN, *Semigroup laws in varieties of solvable groups,* Proc. Cambridge Philos. Soc. **65** (1969) 1–9.

[22] P. LONGOBARDI, M. MAJ AND A.H. RHEMTULLA, *Groups with no free subsemigroups*, Trans. Amer. Math. Soc. **347** (1995) n. 4, 1419–1427.

[23] P. LONGOBARDI AND M. MAJ, *Semigroup identities and Engel groups*, Proc. of Groups St. Andrews in Bath, to appear.

[24] P. LONGOBARDI, M. MAJ AND A.H. RHEMTULLA, *When is a right-orderable group locally indicable?*, Trans. Amer. Math. Soc., to appear.

[25] A.I. MALCEV, *Nilpotent semigroups*, Ivanov Gos. Ped. Inst. Ucen. Zap. Fiz-Mat. Nauk. **4** (1953) 107–111.

[26] B.H. NEUMANN, *On ordered groups*, Amer. J. Math. **71** (1949) 1–18.

[27] B.H. NEUMANN, *An embedding theorem for algebraic systems*, Proc. London Math. Soc. (3) **4** (1954) 138–153.

[28] B.H. NEUMANN AND T. TAYLOR, *Subsemigroups of nilpotent groups*, Proc. Roy. Soc. **A 274** (1963) 1–4.

[29] A. YU. OL'SHANSKII AND A. STOROZHEV, *A group variety defined by a semigroup law*, J. Austral. Math. Soc. (Series A) **60** (1996) 255–259.

[30] J. ROSENBLATT, *Invariant measures and growth conditions,* Trans. Amer. Math. Soc. **197** (1974) 33–53.

[31] A. Shalev, *Combinatorial conditions in residually finite groups II*, J. Algebra **157** (1993) 51–62.

[32] V.M. Tararin, *On the theory of right-orderable groups*, Mat. Zametki, **54** (1993) 96–98.

[33] G.Traustason, *Semigroup identities in 4-Engel groups*, J. Group Theory to appear.

[34] A.A. Vinogradov, *On free product of ordered groups*, Mat. Sb. **67** (N.S. 25) (1949) 163–168.

[35] J.S. Wilson, *Two-generator conditions in residually finite groups*, Bull. London Math. Soc. **23** (1991) 239–248.

[36] E.I. Zelmanov, *On some problems of group theory and Lie algebras,* Math. USSR-Sb. **66** (1) (1990) 159–167.

[37] M. Zorn, *On a theorem of Engel,* Bull. Amer. Math. Soc. **43** (1937) 401–404.