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CATEGORY AND PRODUCTS

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1 Introduction

Recent work by N. Iwase [5] has vastly extended the scope of calculations that can be made concerning Lusternik-Schnirelmann category. In particular, Iwase is able to settle a long standing question raised by Ganea [2], prob.2). Moreover, the method may be used to treat many cases that had been inaccessible. This paper is an exposition of Iwase's work as it pertains to the case $X \times \Sigma A$ with X having L.-S. category 2.

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2 Ganea's Question

The original definition in [6] has undergone extensive reworking. We shall start with the formulation most convenient for Ganea's question, taken from [2].

Let $(X, *)$ be a space with a non-degenerate basepoint. Take $E_0 = E_0(X)$ to be the basepoint and $p_0 : E_0 \to X$ to be the inclusion. For $m \geq 0$ define spaces $F_m = F_m(X)$ and $E_{m+1} = E_{m+1}(X)$ inductively by means of the following pair of push-out and pull-back diagrams,

$$\begin{array}{ccccc} CF_m & \longleftarrow & F_m & \xrightarrow{\ell_m} & PX \\ \downarrow & & \downarrow{\scriptstyle q_m} & & \downarrow \\ E_{m+1} & \underset{j_m}{\longleftarrow} & E_m & \underset{p_m}{\longrightarrow} & X\,. \end{array}$$

Here $PX \to X$ is the path space fibration over X. It does not matter for this paper whether cones or suspensions are reduced, so to fix ideas, we take unreduced cones with vertex at cone coordinate 0. The map $p_{m+1} : E_{m+1} \to X$ extending p_m is obtained by the universal property for push-outs, using the composition

$$CF_m \xrightarrow{C\ell_m} CPX \xrightarrow{e} X\,,$$

with e given by $e(t, \xi) = \xi(t)$.

Following Ganea, we say that cat $X \leq n$ if and only if there is a map

$$s_n : X \to E_n(X)$$

such that $p_n \circ s_n \simeq id$. We write cat $X = n$ for the minimum such n, with $n = \infty$ if no such minimum exists. For spaces of the homotopy type of connected CW complexes, this characterization agrees with the normalized versions of category, which is one less than the original L.-S. formulation. The question raised by Ganea is whether

$$\mathrm{cat}(X \times S^n) = \mathrm{cat}\ X + 1\,.$$

Iwase shows by example that the answer is in the negative.

3 Elementary observations

We note that $F_0 = \Omega X$, $E_1 = \Sigma\Omega X$ and $p_1 : E_1 \to X$ is the evaluation map. The fibration sequence

$$\Omega X \xrightarrow{\partial_m} F_m \xrightarrow{q_m} E_m$$

and the cofibration sequence

$$F_m \xrightarrow{q_m} E_m \xrightarrow{j_m} E_{m+1}$$

are natural for maps $f : X \to Y$. The equations

$$p_m = p_{m+1} \circ j_m\,, \qquad q_{m+1} \circ i_m = j_m \circ q_m \text{ and } \partial_{m+1} = i_m \circ \partial_m$$

hold, where $i_m : F_m \to F_{m+1}$ is defined by naturality for pull-backs. Since the identity map on ΩX factors through Ωp_1, it follows that each boundary map

$$\partial_m : \Omega X \to F_m\,, \qquad m \geq 1 \tag{3.1}$$

is null-homotopic.

Next we recall two elementary facts about maps of pairs. Suppose we have a homotopy commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{\alpha} & Y \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ V' & \xrightarrow[\alpha']{} & Y'\,. \end{array}$$

Each homotopy H from $b\alpha$ to $\alpha' a$ induces a map of pairs for the mapping cones

$$b_H : (C_\alpha, Y) \to (C_{\alpha'}, Y')\,,$$

which extends b. Moreover, there is a homotopy commutative diagram of NDR pairs

$$\begin{array}{ccc} (CV, V) & \xrightarrow{\hat{\alpha}} & (C_\alpha, Y) \\ {\scriptstyle C_a}\downarrow & & \downarrow{\scriptstyle b_H} \\ (CV', V') & \xrightarrow[\hat{\alpha}']{} & (C_{\alpha'}, Y')\,, \end{array} \tag{3.2}$$

with relative homeomorphisms along the horizontal lines.

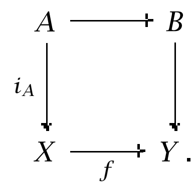
Next, suppose we have a relative homeomorphism of NDR pairs

$$f : (X, A) \to (Y, B)\,,$$

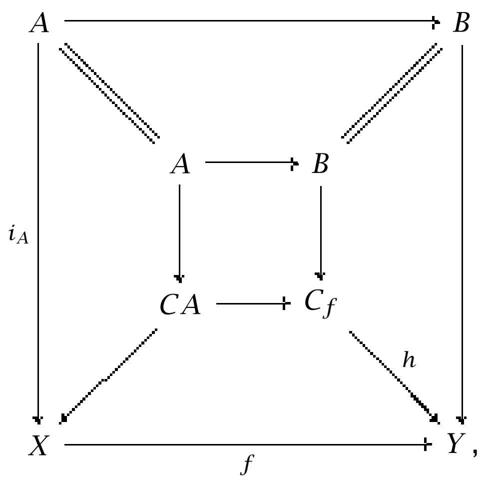
where X is contractible to a point. Then the identity map of B extends to a homotopy equivalence of pairs

$$h : (C_f, B) \to (Y, B)\,. \tag{3.3}$$

To see this, we may use the relative homeomorphism f to place Y in the following push-out diagram,



Let $L : CX \to X$ be a contracting homotopy for X. We obtain h by the universal property for push-outs



where $L \circ C(i_A) : CA \to X$. Since (X, A) is an NDR-pair, the inclusion $A \to CA$ extends to a map $X \to CA$. The compositions

$$X \longrightarrow CA \xrightarrow{C(i_A)} CX \xrightarrow{L} X$$

and

$$CA \xrightarrow{C(i_A)} CX \xrightarrow{L} X \longrightarrow CA$$

are both homotopic, rel A, to identity maps. Thus h is a homotopy equivalence rel B.

4 Category and products

We are concerned with calculations for the category of $X \times \Sigma A$, especially where cat $X = 2$. For this low value some simplifications of Iwase's argument may be made. However, our treatment diminishes the scope of Iwase's approach.

We begin with no assumption on cat X and study a certain subspace W contained in

$$E_2X \times \Sigma A\,.$$

Take $W = E_2X \times \{*\} \cup E_1X \times \Sigma A$.

PROPOSITION 4.1

(a) *cat* $W \leq 2$

(b) *There is a cofibration sequence up to homotopy*

$$F_1X * A \longrightarrow W \xrightarrow{j} E_2X \times \Sigma A\,,$$

with j *inducing a surjection on generalized homotopy groups* $[V, -]$ *where* V *is a suspension.*

PROOF. We have two pushout diagrams:

$$\begin{array}{ccc} F_1 \times \{*\} & \longrightarrow & E_1 \times \{*\} \\ \downarrow & & \downarrow \\ CF_1 \times \{*\} & \longrightarrow & E_2 \times \{*\} \end{array} \tag{4.1}$$

$$\begin{array}{ccc} \Omega X * A & \xrightarrow{w} & E_1 \vee \Sigma A \\ \downarrow & & \downarrow \\ C(\Omega X * A) & \longrightarrow & E_1 \times \Sigma A\,, \end{array} \tag{4.2}$$

where w is the generalized Whitehead product. On the other hand, we may describe W as a push-out by means of the following diagram:

$$\begin{array}{ccc} (E_1 \times \{*\}) \vee (E_1 \vee \Sigma A) & \xrightarrow{\text{fold}} & E_1 \vee \Sigma A \\ \text{inclusion}\big\downarrow & & \big\downarrow \\ (E_2 \times \{*\}) \vee (E_1 \times \Sigma A) & \longrightarrow & W\,. \end{array} \tag{4.3}$$

Composing (4.3) with the one point union of (4.1) and (4.2) presents W in a push-out diagram with a contractible space in one corner. Hence

$$\operatorname{cat} W \leq \operatorname{cat}(E_1 \vee \Sigma A) + 1 = 2\,.$$

For part (b), consider the product of the two relative homeomorphisms

$$(CF_1, F_1) \longrightarrow (E_2, E_1)$$

$$(CA, A) \longrightarrow (\Sigma A, *)\,.$$

We obtain the relative homeomorphism $(CF_1 \times CA, F_1 * A) \longrightarrow (E_2 \times \Sigma A, W)$. Then (b) follows from (3.3). The clause asserting homotopy surjectivity for j follows because W contains the subspace $E_1 \vee \Sigma A$ with this property. □

Suppose X is obtained as a mapping cone C_α for $\alpha : V \to Y$ and we now assume that V is a suspension. Suppose $\operatorname{cat} Y = n - 1 \geq 1$ with structure map

$$s_{n-1} : Y \to E_{n-1} Y\,.$$

PROPOSITION 4.2 *There is a map*

$$H(\alpha) : V \longrightarrow F_{n-1} Y\,,$$

which is unique up to homotopy, such that the following diagram commutes up to homotopy,

$$\begin{array}{ccc} V & \xrightarrow{\alpha} & Y \\ {\scriptstyle F(j)\circ H(\alpha)}\big\downarrow & & \big\downarrow{\scriptstyle E(j)\circ s_{n-1}} \\ F_{n-1}X & \xrightarrow[q_{n-1}]{} & E_{n-1}X\,, \end{array}$$

where j is the inclusion of Y in X. Moreover, there is $s_n : X \to E_n X$ extending the composition $E(j) \circ s_{n-1}$ such that $p_n \circ s_n \simeq id_X$.

PROOF. Let $s : V \to E_1 V \to E_{n-1} V$ be the composition of a structure map for V with the natural inclusion. The compositions of $E(\alpha) \circ s$ and $s_{n-1} \circ \alpha$ with p_{n-1} are homotopic. Since V is a suspension, we may write the difference

$$\delta = s_{n-1} \circ \alpha - E(\alpha) \circ s : V \to E_{n-1} Y\,,$$

and factor it through q_{n-1} by a map

$$H(\alpha) : V \to F_{n-1} Y, \quad q_{n-1} \circ H(\alpha) \simeq \delta\,.$$

It follows from (3.1) that $H(\alpha)$ so defined is unique up to homotopy. By construction, we have the equation

$$q_{n-1} \circ F(j) \circ H(\alpha) \simeq E(j) \circ q_{n-1} \circ H(\alpha) \simeq E(j) \circ \delta \simeq E(j) \circ s_{n-1} \circ \alpha\,.$$

To obtain s_n, we first observe that some map s'_n extending $E(j) \circ s_{n-1}$ exists. The maps $p_n \circ s'_n$ and id_X agree, up to homotopy, when restricted to Y. Thus there is a map $d : \Sigma V \to X$ such that

$$id_X = p_n \circ s'_n + d\,.$$

We have observed (3.1) that p_{n-1} induces an epimorphism on generalized homotopy groups, thus d may be factored through $E_{n-1} X$, we use this factorization to change s'_n to the required map s_n, satisfying the equations

$$j_{n-1} \circ E(j) \circ s_{n-1} = s_n \circ j$$

and

$$p_n \circ s_n \simeq id_X\,.$$

□

Now we impose the restriction that $n = 2$ in Proposition 4.2. We write $\underline{H}(\alpha)$ resp. $\underline{s}_{n-1}$ for the compositions $F(j) \circ H(\alpha)$ resp. $E(j) \circ s_{n-1}$.

PROPOSITION 4.3 *If $\underline{H}(\alpha) * id_A$ is null-homotopic then cat $(X \times \Sigma A) \leq 2$.*

PROOF. First we combine Proposition 4.2 with (3.2) to obtain the following homotopy commutative diagram:

$$\begin{array}{ccc} (CV, V) & \xrightarrow{\hat{\alpha}} & (X, Y) \\ {\scriptstyle C\underline{\mathrm{H}}(\alpha)}\downarrow & & \downarrow{\scriptstyle s_2} \\ (CF_1X, F_1X) & \xrightarrow[\hat{q}_1]{} & (E_2X_1E_1X)\,. \end{array}$$

Next we take the product of the maps in this diagram with the relative homeomorphism

$$(CA, A) \to (\Sigma A, *)\,.$$

The product of relative homeomorphisms is a relative homeomorphism, hence we may apply (3.3) to the rows and obtain the following diagram:

$$\begin{array}{ccccc} V * A & \longrightarrow & X \times \{*\} \cup Y \times \Sigma A & \longrightarrow & X \times \Sigma A \\ {\scriptstyle \underline{\mathrm{H}}(\alpha) * id_A}\downarrow & & \downarrow & & \downarrow{\scriptstyle s_2 \times id} \\ F_1X * A & \longrightarrow & W & \xrightarrow[j]{} & E_2X \times \Sigma A\,, \end{array}$$

where the left square is homotopy commutative and the right square is strictly commutative. Moreover, the rows are cofibration sequences, up to homotopy.

The hypothesis on $\underline{\mathrm{H}}(\alpha)$ yields a map

$$\lambda : X \times \Sigma A \longrightarrow W$$

extending the middle vertical map. Moreover $j \circ \lambda$ and $s_2 \times id$ agree on the subspace

$$X \times \{*\} \cup Y \times \Sigma A\,.$$

Since j induces an epimorphism on generalized homotopy groups, we may choose λ satisfying

$$j \circ \lambda \simeq s_2 \times id$$

as well as extending the middle vertical map. Taking composition with $p_2 \times id_{\Sigma A}$, we see that W dominates $X \times \Sigma A$, completing the proof. □

5 Iwase's examples

We now turn to Iwase's examples. These are of the form $Q \times S^n$ where Q is a certain two-cell complex. Let $\alpha : S^r \to S^m$ be a map of spheres with mapping cone Q. It is well known [1] that cat $Q = 1$ if and only if α is a co-H-map. Moreover, the issue whether α is a co-H-map is settled by the Hilton-Hopf invariants in the formula ([9], p.533).

$$(\iota_1 + \iota_2) \circ \alpha = \iota_1 \alpha' + \iota_2 \circ \alpha'' + \sum w \circ h_w(\alpha),$$

where w ranges over a Hall basis for the free Lie algebra on two generators ι_1, ι_2 and

$$w = w(\iota_1, \iota_2) : S^{km+1} \longrightarrow S^m \vee S^m$$

is a Whitehead product of weight k. The map

$$h_w : \pi_r S^m \longrightarrow \pi_r S^{km+1}$$

is the Hilton-Hopf invariant associated to w.

If m is odd and α has order equal to a power of an odd prime p, then

$$h_w(\alpha) = 0$$

provided the weight of w is not congruent to 1 mod $p - 1$. Moreover, if $h_w(\alpha) = 0$ for w of weight p, then all $h_w(\alpha) = 0$. These facts are proved in [4]. The latter condition is equivalent to α belonging to the kernel of the p-th James-Hopf invariant

$$H_p : \Omega S^{2n+1} \to \Omega S^{2np+1}, \qquad m = 2n + 1.$$

We now look at some specific elements. Let α_1 in $\pi_6 S^3$ be an essential element of order 3. Then α_1 is a co-H-map for dimensional reasons. The classical Hopf maps

$$\eta : S^3 \longrightarrow S^2, \qquad \sigma : S^{15} \longrightarrow S^8$$

have Hopf invariant one, with

$$(\iota_1 + \iota_2) \circ \eta = \iota_1 \circ \eta' + \iota_2 \circ \eta'' + [\iota_1, \iota_2]$$

and likewise for σ. Now consider the composition

$$\beta_1 = \eta \circ \alpha_1 \circ \Sigma^3 \alpha_1 : S^9 \longrightarrow S^2 ,$$

which is a composition of η with a co-H-map. This composition has non-trivial Hilton-Hopf invariant

$$\alpha_1 \circ \Sigma^3 \alpha_1 .$$

Hence $Q_1 = S^2 \cup_{\beta_1} e^{10}$ has category = 2.

We shall consider two more similar examples. The Whitehead product

$$[\iota_{15}, \iota_{15}] : S^{29} \longrightarrow S^{15}$$

is both essential and a suspension. Thus the composition

$$\beta_2 = \sigma \circ [\iota_{15}, \iota_{15}]$$

has non-trivial Hilton-Hopf invariant equal to $[\iota_{15},\ \iota_{15}]$. The essential element of order p constructed by Moore [7]

$$w_n : S^{2np-3} \longrightarrow S^{2n-1}$$

for $n \geq 1$ and p any odd prime has p-th James-Hopf invariant 0 for dimensional reasons. The fact that this map is essential is the mod p Hopf invariant one result. We take n even and consider

$$\beta_3 = [\iota_n, \iota_n] \circ w_n : S^{2np-3} \longrightarrow S^n .$$

This composition has non-trivial Hilton-Hopf invariant equal to $2w_n$. Thus each of the mapping cones of β_i, Q_i $i = 1, 2, 3$ has category = 2.

Theorem 5.1 (Iwase's Theorem) *cat* $(Q_i \times S^n) = 2$ *for* $n \geq 2$ *and* $i = 1, 3$. *cat* $(Q_2 \times S^n) = 2$ *for* $n \geq 1$.

Proof. The fundamental theory developed by Ganea [3] provides two homotopy commutative diagrams,

$$\begin{array}{ccc} S^m & \xrightarrow{\text{pinch}} & S^m \vee S^m \\ \Big\downarrow {\scriptstyle \underline{s}_1} & & \Big\downarrow \\ E_1 Q & \xrightarrow[s]{} & Q \vee Q , \end{array}$$

with S inducing a monomorphism on generalized homotopy groups and

$$\begin{array}{ccccc}
S^{km+1} & \xrightarrow{w} & S^m \vee S^m & \longrightarrow & Q \vee Q \\
\Big\| & & & & \Big\uparrow {\scriptstyle S\circ q_1} \\
S^{km+1} & & \xrightarrow[\underline{w}]{\hspace{6em}} & & F_1 Q
\end{array}$$

for each Whitehead product w of weight k. Moreover, $\underline{w}$ is unique up to homotopy. Hence we may express the total Hopf invariant of $\alpha : S^r \to S^m$ in terms of the Hilton-Hopf invariants and Whitehead products

$$S^r \longrightarrow \bigvee_w S^{km+1} \longrightarrow F_1 Q$$

$$\underline{H}(\alpha) = \bigoplus_w (\underline{w} \circ h_w(\alpha)) \,.$$

For $\beta_i, i = 1, 3$, the double suspension of the total Hopf invariant is 0, while for β_2, a single suspension achieves this. Hence Theorem 5.1 follows from Proposition 4.3. □

We conclude this section by comparing Iwase's argument with Proposition 4.1, part (a). Iwase makes use of the tensor product filtration ([8], p. 358) to obtain an analogue with $E_k X, E_{k-1} X$ in place of $E_2 X, E_1 X$. An illustration of the wider scope for Iwase's method is provided by the following example. For $\alpha : S^r \to S^m$, we have the following homotopy commutative diagrams:

$$\begin{array}{ccc}
(CS^r, S^r) & \longrightarrow & (Q, S^m) \\
\Big\downarrow & & \Big\downarrow \\
(CF_1 Q, F_1(Q)) & \longrightarrow & (E_2 Q, E_1 Q) \,.
\end{array}$$

As in the proof of Proposition 4.1, part (b), we may take the product of this diagram with itself and extract a ladder of cofibration sequences. The result is a homotopy commutative diagram

$$\begin{array}{ccccc} S^r * S^r & \longrightarrow & Q \times S^m \cup S^m \times Q & \longrightarrow & Q \times Q \\ \downarrow{\scriptstyle \underline{H}(\alpha) * \underline{H}(\alpha)} & & \downarrow & & \downarrow \\ F_1Q * F_1Q & \longrightarrow & E_2Q \times E_1Q \cup E_1Q \times E_2Q & \longrightarrow & E_2Q \times E_2Q\,, \end{array}$$

with a factorization in the right hand square just as in that argument. The tensor product filtration provides a map

$$E_2Q \times E_1Q \cup E_1Q \times E_2Q \longrightarrow E_3(Q \times Q)$$

and it follows that cat $(Q \times Q) \leq 3$, for the cases considered in Proposition 5.1. This is an easy argument to make, but one unknown to this writer, before Iwase's work.

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