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**SMALL DENOMINATORS AND EXPONENTIAL STABILITY:
FROM POINCARÉ TO THE PRESENT TIME**

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ABSTRACT. The classical problem of small denominators is revisited in its historical development, ending with recent results on exponential stability.

1 Overview

In 1959 J.E. Littlewood published two papers [30, 31] devoted to the stability of the Lagrangian equilateral equilibria of the problem of three bodies. Let me quote the *incipit* of the first of those papers:

The configuration is one of two point masses S and J , with $m_J = \mu m_S$, and a body P of zero mass, the law of attraction being the inverse square; the equilateral triangle SJP , of side 1, rotates in equilibrium with unit angular velocity about the centre of gravity of S and J . If $\mu < \mu_0$, where μ_0 is the smaller root of $\mu_0(1 + \mu_0)^{-2} = \frac{1}{27}$, the configuration is stable in the sense that a small disturbance does not double in a few revolutions; there are real periods $2\pi/\lambda_1, 2\pi/\lambda_2$ of 'normal oscillations'.

It is notorious that no non-trivial system whatever is known to be stable (or bounded) over infinite time. It is possible, however, to ask a less far-reaching question: given that the initial disturbance in coordinates and velocities is of order ε , for how long a time, in terms of ε , can we say the disturbance will remain of order ε ? We shall find that, for almost all values of $\mu < \mu_0$, this time is as long as $\exp(A\varepsilon^{-1/2}|\log \varepsilon|^{-3/4})$, where A depends only on μ ; while not eternity, this is a considerable slice of it.

Following Littlewood, I will call *exponential stability* the property of a system of being stable for a time that increases exponentially with the inverse of a perturbation parameter ε .

The problem of the Lagrangian equilibria is just the simplest case of the many stability problems that show up in the dynamics of a many body system, like, e.g., the solar system. By the way, Littlewood himself in a note to his papers says that he started his investigations without being aware of previous works on the same subject, e.g., by Whittaker and Birkhoff, that he discovered only later. On the other hand, Littlewood's work appeared while a significant progress on the general subject of the dynamics of nearly integrable Hamiltonian systems was starting. A few years before Kolmogorov [25] had announced his celebrated theorem on persistence of invariant tori under small perturbations; his work was going to be extended a few years later by Moser [37] and Arnold [1], thus originating what is now called *KAM theory*. Furthermore, the phenomenon of exponential stability had already been investigated by Moser [36].

As a matter of fact, the problem of Lagrangian equilibria investigated by Littlewood is a very classical one, being related to the problem of the so called *small denominators* that was well known to people interested in the dynamics of the planetary system.

My purpose here is to illustrate in an informal manner the theory of exponential stability and its relations with the problem of small denominators. I will start with a simple example that illustrates how the problem of small denominators arises, and how it is related to the problem of *secular terms* that was a major one in the planetary theories of the past century. Then I will follow the historical de-

velopment of the theory. However, since this is not intended as an historical note, I will emphasize only the steps that have shown to be relevant, according to my limited experience and knowledge. Finally, I will conclude by illustrating how the theory of exponential stability may be pushed up to the point of giving results that are realistic for a physical model. To this end, I will use as example the problem of stability of the orbits of the Trojan asteroids that are close to the equilateral solutions of Lagrange in the Sun–Jupiter system.

2 Model problems

I will consider three typical model problems that often appear in the investigation of the dynamical behaviour of physical systems.

The first model is a harmonic oscillator of proper frequency λ subjected to a nonlinear, time dependent forcing that is periodic with frequency ν . The corresponding differential equation may be written in the general form

$$\begin{aligned}\ddot{x} + \lambda^2 x &= \psi_0(\nu t) + x\psi_1(\nu t) + x^2\psi_2(\nu t) + \cdots \\ \psi_j(\nu t) &= \psi_j(\nu t + 2\pi), \quad j = 0, 1, \dots\end{aligned}\quad (2.1)$$

with $x \in \mathbb{R}$ and the functions $\psi_j : \mathbb{T} \rightarrow \mathbb{R}$ will be assumed to be real analytic. This model has been investigated in particular by Lindstedt [27, 28, 29], who introduced his method for constructing solutions with given frequencies. I will discuss this method in some detail in the next section, making reference to the particular case of the *Duffing equation*

$$\ddot{x} + x = \varepsilon(\cos \nu t + x^3), \quad (2.2)$$

where ε is a (small) perturbation parameter. This is perhaps the simplest example of a non-integrable system exhibiting all problems due to the small denominators. Remark that forgetting the periodic forcing $\cos(\nu t)$ the resulting system may be integrated with elementary methods, being just a nonlinear oscillator; similarly, forgetting the cubic term x^3 one obtains a forced linear oscillator, which is trivially integrable, too.

The second model is a system of canonical differential equations in the neighborhood of an elliptic equilibrium point. In general the Hamiltonian may be given the form of a system of n harmonic oscillators with a nonlinear perturbation, namely

$$H(x, y) = \frac{1}{2} \sum_{l=1}^n \omega_l (x_l^2 + y_l^2) + H_3(x, y) + H_4(x, y) + \dots \quad (2.3)$$

where $(x, y) \in \mathbb{R}^{2n}$ are the canonically conjugated variables, $\omega \in \mathbb{R}^n$ is the vector of frequencies (that are assumed to be non vanishing), and $H_s(x, y)$ for $s > 2$ is a homogeneous polynomial of degree s in (x, y) . The problem of the Lagrangian equilibria investigated by Littlewood may be given this form. It will be useful to rewrite the Hamiltonian (2.3) in action-angle variables $(p, q) \in [0, \infty)^n \times \mathbb{T}^n$, introduced via the canonical transformation

$$x_j = \sqrt{2p_j} \cos q_j, \quad y_j = \sqrt{2p_j} \sin q_j. \quad (2.4)$$

The Hamiltonian then reads

$$H(p, q) = \sum_{l=1}^n \omega_l p_l + H_3(p^{1/2}, q) + H_4(p^{1/2}, q) + \dots \quad (2.5)$$

where H_s turns out to be a trigonometric polynomial of degree s in the angles q .

The third problem is a canonical system of differential equations with Hamiltonian

$$H(p, q, \varepsilon) = h(p) + \varepsilon f(p, q, \varepsilon), \quad (2.6)$$

where $(p, q) \in \mathcal{G} \times \mathbb{T}^n$ are action angle variables, \mathcal{G} being an open set, and ε is a perturbation parameter. The functions $h(p)$ and $f(p, q, \varepsilon)$ are assumed to be real analytic functions of (p, q, ε) for ε in a neighborhood of the origin. This problem has been called by Poincaré *le problème général de la dynamique* ([42], tome I, chap. I, § 13). Indeed, it was known from a long time that the Hamiltonian of some interesting systems that are integrable in Liouville's sense* may be

*That is, the system possesses n independent first integrals that are in involution. According to Liouville's theorem the system may be integrated by quadratures.

written in action-angle variables p, q as a function $h(p)$ independent of the angles. Classical examples are the Keplerian problem, the Hamiltonian of the solar system when the mutual interaction of the planets is neglected and the rigid body with a fixed point. The recent theorems of Arnold and Jost state that action-angle variables may be introduced for an integrable Hamiltonian system under quite general conditions.

The dynamics of the integrable system with $H = h(p)$ is quite simple. The canonical equations are

$$\dot{q}_j = \frac{\partial h}{\partial p_j}(p) =: \omega_j(p), \quad \dot{p}_j = 0, \quad j = 1, \dots, n. \quad (2.7)$$

Denoting by $p^{(0)}, q^{(0)}$ the initial point for $t = 0$ the trivial solution is

$$q(t) = q^{(0)} + \omega(p^{(0)}) t, \quad p(t) = p^{(0)}. \quad (2.8)$$

Hence the orbits lie on invariant tori \mathbb{T}^n parameterized by the actions p . The system (2.2) for $\varepsilon = 0$ and the system (2.3) for $H_3 = H_4 = \dots = 0$ are of this form.

3 The problem of small denominators

I will illustrate the problem by making reference to the simple system (2.2). However, the discussion below applies with straightforward modifications to the cases of the equation (2.1) and of the Hamiltonian system (2.3). The more general problem of the Hamiltonian system (2.6) requires some different setting in order the method be applicable. This is widely discussed by Poincaré in [42], tome II, chap. IX.

In view of the fact that for $\varepsilon = 0$ the system (2.2) is trivially integrable, a natural attempt is to look for a solution as a power expansion in the small parameter ε , namely

$$x(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \dots, \quad (3.1)$$

where the functions $x_0(t), x_1(t), x_2(t), \dots$ are the unknown functions to be determined. By substitution in eq. (2.2) and by comparison of coefficients of the same power of ε we get the infinite system

of equations

$$\begin{aligned}
 \ddot{x}_0 + x_0 &= 0, \\
 \ddot{x}_1 + x_1 &= \cos(\nu t) + x_0^3, \\
 &\dots \\
 \ddot{x}_s + x_s &= \psi_s(x_0, \dots, x_{s-1}), \\
 &\dots
 \end{aligned} \tag{3.2}$$

where ψ_s is a known function of x_0, \dots, x_{s-1} only (actually a third degree polynomial in our case). We may attempt at solving this infinite system recursively. Forgetting an unessential initial phase, a solution of the equation for x_0 is $x_0(t) = a \cos t$, where a is the amplitude of the unperturbed oscillation, an arbitrary parameter to be determined by the initial conditions. Replacing the expression above of $x_0(t)$ in the r.h.s. of the equation for $x_1(t)$ we get the non homogeneous linear equation of a forced oscillator

$$\ddot{x}_1 + x_1 = \cos(\nu t) + \frac{a^3}{4} \cos(3t) + \frac{3a^3}{4} \cos t.$$

The solution is

$$x_1(t) = \frac{1}{1 - \nu^2} \cos(\nu t) - \frac{a^3}{32} \cos(3t) + \frac{a^3}{8} t \sin t;$$

we do not need to add the arbitrary solution of the homogeneous equation. Proceeding the same way, we easily see that the r.h.s. of the equation for x_s will be a known trigonometric polynomial in t with coefficients that are polynomials in t . Terms that have coefficients t, t^2, \dots are traditionally named *secular terms*, because in the case of the planetary motions they correspond to a slow drift, e.g., of the semi-major axes or of the eccentricities of the orbits; the effect of such a drift is a deviation from the simple Keplerian motion on elliptic orbits, and may be observed by collecting data over a few centuries. The appearance of secular terms in the solutions raises a number of problems.

A first problem concerns the *stability* of the motion. For instance, suppose that the eccentricity of the Earth's orbits is really subjected to an uniform secular drift; then it could, e.g, increase so as to take

a value that is significantly larger than the current one, thus making the Earth to approach the Sun quite closely when it reaches the perihelion of the orbit. This would be incompatible with the existence of life on the Earth. Similarly, a uniform drift of the eccentricities and semi-major axes of all planets could cause two planets to collide, or one planet to escape from the solar system. Remark that this could happen in a time that is quite short with respect to the estimated age of the solar system. Thus, it seems likely to expect that the contributions of the secular terms do compensate each other, so that the overall effect could be bounded.

A second problem is that the solution constructed above is only *local* in time. That is, for a fixed ε the series is expected in general to be convergent only for t small enough; hence, the solution would be valid only for a quite short time, thus making the problem, e.g., of calculating the orbits of the planets for a quite long time a very difficult one. On the other hand, looking at the equation (2.2) one would rather expect to be able to write the solution as a superposition of periodic motions. This raises two classical problems, namely:

- (i) to write the general solution of eq. (2.1) (or, more generally, of the equations for the n -body problem) as a power series in ε that is uniformly convergent for all times;
- (ii) to write the coefficients of the power series in ε as trigonometric functions of t , still keeping the property of uniform convergence as in (i).

The problem (i), referred to the case of the planetary system, was proposed as a prize question sponsored by the king of Sweden. The prize was awarded to Poincaré, who did not actually solve the problem, but presented a memoir containing a wealth of new ideas. A revised version of the memoir was later published in [41].

The problem (ii) is partially solved via the method of Lindstedt [27, 28, 29]. We look for a *single* solution which is a perturbation of an harmonic oscillation with a *given* frequency $\omega = 1 + \varepsilon\delta$ (recall that 1 is the proper frequency of the unperturbed oscillator). To this end, let us rewrite eq. (2.2) as

$$\ddot{x} + \omega^2 x = \varepsilon(\cos(\nu t) + x^3 - \delta x). \quad (3.3)$$

We look again for a solution as a power series in ε of the form (3.1). This gives for $x_0(t)$ the equation

$$\ddot{x}_0 + \omega^2 x_0 = 0,$$

with solution (still forgetting an unessential initial phase)

$$x_0(t) = a_0 \cos(\omega t),$$

where a_0 is left arbitrary. By substitution in (3.3) we get for x_1 the equation

$$\ddot{x}_1 + \omega^2 x_1 = \cos(\nu t) + \frac{a_0^3}{8} \cos(3\omega t) + \frac{3a_0^3}{8} \cos(\omega t) - \delta a_0 \cos(\omega t).$$

Now, the terms $\cos(\omega t)$ would produce again secular terms in the solution. We avoid them by determining the amplitude a_0 so that these terms do disappear, i.e., as the (positive) solution of the equation

$$\frac{3a_0^2}{8} - \delta = 0.$$

This means that having fixed the frequency ω we are forced to select a single solution with amplitude a_0 depending on the frequency. Having so determined a_0 , the solution of the equation for x_1 is

$$x_1(t) = \frac{1}{\omega^2 - \nu^2} \cos(\nu t) - \frac{a_0^3}{32\omega^2} \cos(3\omega t) + a_1 \cos(\omega t + \varphi_1),$$

where a_1 and φ_1 are left arbitrary; these constants will be determined so as to remove all secular terms from the equation for x_2 . However, the denominator $\omega^2 - \nu^2$ forces us to exclude the frequency $\omega = \pm\nu$. A moment's thought will allow us to realize that the term x_s of the solution will be determined as a sum of trigonometric terms in $(j\omega + k\nu)t$, with divisors of the form $j\omega + k\nu$, where j, k are arbitrary integers. Thus, the following problems arise:

- (i) even disregarding the convergence of the series so produced, we must exclude all values of ω such that ω/ν is a rational number;

- (ii) the denominators $j\omega + k\nu$, although non vanishing by the condition above, may become arbitrarily small; this raises serious doubts on the convergence of the series.

The latter is actually the simplest aspect of the so called *problem of small denominators*.

It should be mentioned that the problem of the convergence of Lindstedt's series has been accurately investigated by Poincaré ([42], tome II, chap. XIII, § 148–149). However, he was unable to decide this question (here, n_1, n_2 play the role of our frequencies ω, ν):

... les séries ne pourraient-elles pas, par exemple, converger quand ... le rapport n_1/n_2 soit incommensurable, et que son carré soit au contraire commensurable (ou quand le rapport n_1/n_2 est assujetti à une autre condition analogue à celle que je viens d' énoncer un peu au hasard)?

Les raisonnements de ce chapitre ne me permettent pas d' affirmer que ce fait ne se présentera pas. Tout ce qu' il m' est permis de dire, c' est qu' il est fort invraisemblable.

The story of Lindstedt's series does not end here. The challenging question of the convergence was *indirectly* solved in 1954 by Kolmogorov [25]. His theorem implies that the series of Lindstedt are uniformly convergent in time provided the frequencies ω, ν satisfy some condition of strong non-resonance that include the case suggested "un peu au hasard" by Poincaré. Kolmogorov's method is based on a scheme of fast convergence (called by Kolmogorov *generalized Newton's method*, and often referred to as *quadratic method*) that avoids the classical expansions in a perturbation parameter. It is often said that Kolmogorov announced his theorem without publishing the proof; as a matter of fact, his short communication contains a sketch of the proof where all critical elements are clearly pointed out. Detailed proofs were published later by Moser [37] and Arnold [1]; the theorem become thus known as KAM theorem. The indirect proof of convergence of Lindstedt's series via Kolmogorov's method is discussed in a paper by Moser [38]; however, he failed to obtain a *direct* proof based, e.g., on Cauchy's classical method of majorants applied

to Lindstedt's expansions in powers of ε . As discovered by Eliasson [11], this is due to the presence in Lindstedt's classical series of terms that grow too fast, due precisely to the small denominators, but are canceled out by internal compensations (this was written in a report of 1988, but was published only in 1996). Explicit constructive algorithms taking compensations into account have been recently produced by Gallavotti, Chierchia, Falcolini, Gentile and Mastropietro (see, e.g., [12, 9, 13] and the references therein). An explicit algorithm that avoids the need of compensations has been recently introduced by Locatelli and the author [21, 22].

It is not my purpose here to go further into the KAM theory: the number of papers is so big that an exhaustive report could not fit into the present note. I just recalled a few works that have some direct relation with Lindstedt's series. However, let me emphasize that the beautiful results of KAM theory do not represent a complete solution of the problem (ii) above. Indeed, Lindstedt's series can not be constructed for arbitrary initial data, but only for a set of initial data of large measure corresponding to invariant tori filled up by quasiperiodic motions; the resonant frequencies have been excluded, according to the point (i) above; moreover, frequencies that are too close to resonance must be excluded, too.

In order to illustrate this matter at least at a phenomenological level let me consider the simple case of a Hamiltonian system of the form (2.4), namely

$$H = \frac{\omega_1}{2}(x_1^2 + y_1^2) - \frac{\omega_2}{2}(x_2^2 + y_2^2) + x_1^2 x_2 - \frac{1}{3}x_2^3, \quad (3.4)$$

with $\omega_1 = 1$ and $\omega_2 = (\sqrt{5} - 1)/2$ (the golden number). Figure 1 represents the Poincaré section of different orbits with the surface $x_1 = 0$. All orbits lie on the energy surface $H(x, y) = 0.015$, and the points are represented in the plane (x_2, y_2) .[†] The central point in the

[†]Having fixed the value E of the energy and the surface of section $x_1 = 0$, a point (x_2, y_2) determines the unique initial point $(0, x_2, y_1, y_2)$ in phase space, where $y_1 > 0$ is the positive solution of the equation $H(0, x_2, y_1, y_2) = E$. The orbit is integrated numerically until it intersects again the surface $x_1 = 0$ with the condition $y_1 > 0$. Thus, a one to one map of the plane (x_2, y_2) into itself is generated, which is named the Poincaré section. Figure 1 represents several orbits

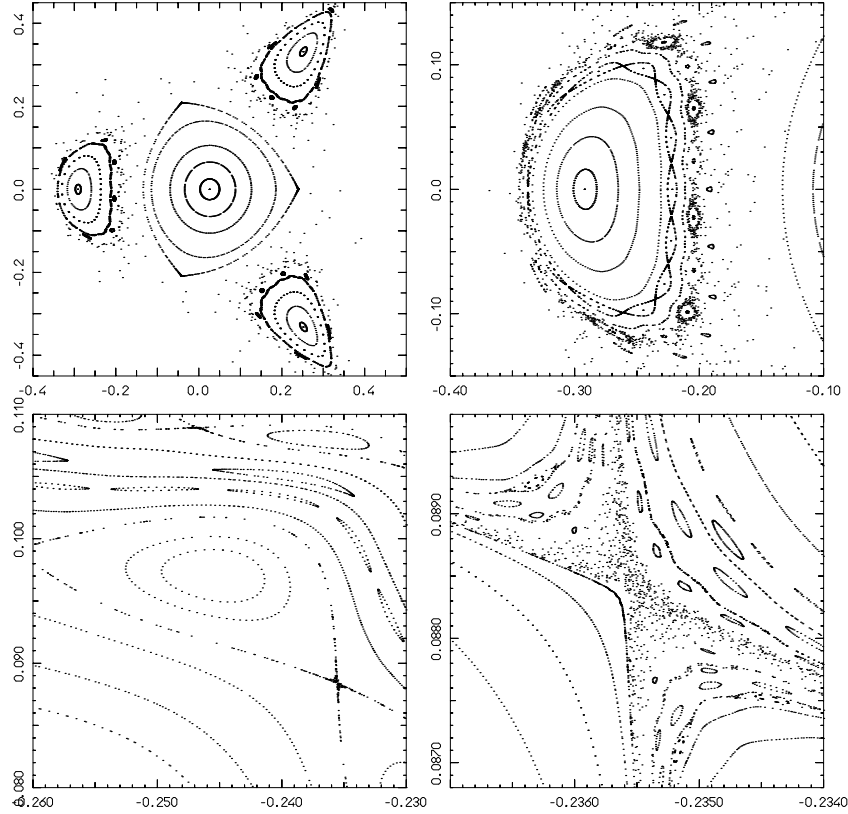


Figure 1: *Poincaré section for the Hamiltonian system (3.4) with $\omega_1 = 1$ and $\omega_2 = (\sqrt{5} - 1)/2$ on the energy surface $H(x, y) = 0.015$. The surface of section is $x_1 = 0$, and the figure represents the projection of the energy surface on the plane (x_2, y_2) . The complexity of the orbits is illustrated by the successive enlargements of parts of the first figure.*

first figure (upper left corner) represents a periodic orbit that intersects the surface of section always at the same point, thus appearing as a fixed point in the Poincaré section. The points that appear to lie

generated by the map, corresponding to different initial points. Remark that the figure is actually a projection of the energy surface on the plane (x_2, y_2) . For more details, including a thorough illustration of the phenomenology, an excellent reference is the celebrated paper of Hénon and Heiles [24].

on circle-shaped curves close to the central point represent orbits that in the approximation of the figure look as quasiperiodic orbits lying on invariant tori, with frequencies depending on the torus.[‡] The invariant curves appear to be broken by the perturbation when the frequencies approach a resonant value. In our case the strongest resonance corresponds to $\omega_2/\omega_1 \sim 2/3$, which is a low order approximation of the golden number. The main effect of this resonance is the creation of the three regions with closed invariant curves that are separated from the central region. At the center of the three regions there are three points that represent the section of a periodic orbit of period 3 (that is, the orbit comes back to the initial point after three intersections with the surface $x_1 = 0$). The points scattered at random belong to a single chaotic orbit that eventually escapes to infinity. The remaining three figures are successive enlargements of small zones of the first figure that give a rough idea of the complicated global behaviour of the orbits, due to resonances: the creation of periodic orbits surrounded by invariant closed curves appears repeatedly as the resolution increases. As a matter of fact only a few periodic orbits with their accompanying invariant curves may be evidenced in the figure. For, the size of the region influenced by a resonance becomes exceedingly small as the order of the resonance increases.

4 Formal construction of first integrals

Let us now consider the general Hamiltonian system (2.6), writing more explicitly the expansion of the perturbation $f(p, q, \varepsilon)$ in powers of ε as

$$H(p, q) = h(p) + \varepsilon f_1(p, q) + \varepsilon^2 f_2(p, q) + \dots ; \quad (4.1)$$

[‡]Consider for a moment only the quadratic part of the Hamiltonian. Then the system is trivially integrable, and the orbits lie on invariant tori that are the Cartesian product of circles in the planes (x_1, y_1) and (x_2, y_2) , respectively. The motion on these tori is quasiperiodic with frequencies that coincide with the unperturbed frequencies ω_1, ω_2 . The cubic terms in the Hamiltonian introduce a non-linearity that makes the frequencies to depend on the initial data, thus breaking the simple behaviour of the unperturbed system.

recall that all functions are assumed to be analytic. In view of Liouville's theorem, we may try to find n independent first integrals that are in involution. Precisely, we look for first integrals of the form

$$\Phi(p, q) = \Phi_0(p, q) + \varepsilon\Phi_1(p, q) + \varepsilon^2\Phi_2(p, q) + \dots \quad (4.2)$$

by trying to solve the equation $\{H, \Phi\} = 0$, where $\{\cdot, \cdot\}$ denotes the Poisson bracket. By substituting the expansions of H and Φ above and comparing the coefficients of the same power of ε we get the infinite system

$$\begin{aligned} \partial_\omega \Phi_0 &= 0 \\ \partial_\omega \Phi_1 &= -\{f_1, \Phi_0\} \\ \partial_\omega \Phi_2 &= -\{f_1, \Phi_1\} - \{f_2, \Phi_0\} \\ &\dots \end{aligned} \quad (4.3)$$

where $\partial_\omega \cdot = \{h(p), \cdot\} = \sum_l \omega_l(p) \frac{\partial \cdot}{\partial q_l}$. The equation for Φ_0 is a trivial one, since it is certainly satisfied by any of the action variables p_j . As Poincaré proves, if the unperturbed Hamiltonian is non degenerate, i.e., if

$$\det \left(\frac{\partial^2 h}{\partial p_i \partial p_j} \right) \neq 0, \quad (4.4)$$

then Φ_0 needs to be independent of the angles q . At higher orders in ε we must solve an equation of the form

$$\partial_\omega \Phi = \Psi, \quad (4.5)$$

where $\Psi(p, q)$ is a known function that is supposed to be periodic in the angles q and Φ is to be determined with the same periodicity condition. The problem of solving an equation of this type is a common one in perturbation theory. Using the Fourier expansion

$$\Psi(p, q) = \sum_{k \in \mathbb{Z}^n} c_k(p) \exp(\langle k, q \rangle)$$

with known coefficients $c_k(p)$, the formal solution for Φ is

$$\Phi(p, q) = -i \sum_{k \in \mathbb{Z}^n} \frac{c_k(p)}{\langle k, \omega(p) \rangle} \exp(\langle k, q \rangle); \quad (4.6)$$

however, such a solution may be accepted only if $c_k(p) = 0$ whenever $\langle k, \omega(p) \rangle = 0$. This raises two further problems:

- (i) for $k = 0$ we must have $c_k(p) = 0$, i.e., the average of $\Psi(p, q)$ over the angles must vanish;
- (ii) for $k \neq 0$ and under the non degeneracy condition (4.4) the set of points p where some denominator $\langle k, \omega(p) \rangle$ vanishes is dense in the action domain.

Let us skip (i) for a moment (it is satisfied in the equation for Φ_1). Condition (ii) is quite delicate, since it imposes very strong constraints on the coefficients $c_k(p)$. Poincaré's conclusion is the following

THEOREM 4.1 *Generically, a Hamiltonian of the form (4.1) satisfying the non degeneracy condition (4.4) does not possess any first integral of the form (4.2) independent of H .*

For a proof, see [42], tome I, chap. V; the same proof is reported in [49], chap. XIV, § 165.

It was soon realized by Whittaker that the difficulty of constructing formal first integrals is by far less acute if one considers the case of an elliptic equilibrium point, where the Hamiltonian may be given the form (2.3) (or (2.5) in action-angle variables). Indeed, the unperturbed Hamiltonian $h(p) = \sum_l \omega_l p_l$ is degenerate, so that one of the hypotheses of Poincaré's theorem is not fulfilled, and moreover the frequencies ω are independent of the actions p . Therefore, the denominators in the solution (4.6) of the equation for a first integral do not vanish provided the frequencies ω satisfy the *non resonance condition*

$$\langle k, \omega \rangle \neq 0 \quad \text{for } 0 \neq k \in \mathbb{Z}^n. \quad (4.7)$$

Hence, the construction is formally consistent provided the condition (i) above is fulfilled, namely if the r.h.s. of the equations (4.3) for the first integrals has zero average over the angles. This looks as a trivial problem, the solution of which has revealed to be surprisingly difficult.[§] Up to my knowledge, the most general direct proof of the consistency of the formal construction is the following

[§]As an historical remark, it is curious that Whittaker in the very exhaustive paper [48] did not mention the problem. A few years later Cherry wrote two

THEOREM 4.2 *Let the Hamiltonian (2.3) be even in the momenta y , i.e., $H(x, y) = H(x, -y)$, and let the frequencies satisfy the non-resonance condition (4.7). Then the system possesses n formal first integrals*

$$\Phi^{(l)}(x, y) = \frac{x_l^2 + y_l^2}{2} + \Phi_3(x, y) + \Phi_4(x, y) + \dots, \quad l = 1, \dots, n, \tag{4.8}$$

that are independent and in involution, the functions $\Phi_s(x, y)$ for $s \geq 3$ being homogeneous polynomials of degree s .

The condition $H(x, y) = H(x, -y)$ means that the system is reversible.[¶] The surprisingly simple argument is that, due to the parity conditions, the r.h.s. of the equations (4.3) turn out to be odd functions of y . On the other hand, the average depends only on the action variables p , and so it is an even function of y ; hence it must vanish. For a complete proof see [10].

The theorem above implies that the system is formally integrable by quadratures; moreover, there are action-angle variables so that the solutions may be written as series that are trigonometric in t .^{||} This looks as a complete answer to the classical problem of finding

papers [7, 8] where a lot of work is devoted to this problem, but without reaching a definite conclusion. An indirect solution was found by Birkhoff, using the method of normal form that goes usually under his name ([4], chap. III, § 8). As a matter of fact, this problem of consistency is strictly related to a similar problem that shows up in Lindstedt's method (and that I did not mention before). The latter problem is solved by Poincaré precisely by going through a transformation of the Hamiltonian into Birkhoff's normal form ([42], tome II, chap.VIII, § 123-125).

[¶]If one removes the parity condition then the consistency of the construction follows quite easily from the existence of Birkhoff's normal form, but no direct argument has been found, up to my knowledge. Things are even worse if one allows the frequencies to be resonant. For a discussion of this problem see [14]. In the case of Lindstedt's series a similar argument based on parity has been used by Gallavotti [12].

^{||}See [49], chap. XVI, § 199, in particular the last sentence. It is worth noting that the procedure outlined by Whittaker is actually the construction of action variables described, e.g., in Born's book [5]. Precisely, the invariant surfaces defined by the first integrals are diffeomorphic to n -dimensional tori, and the action variables are determined by calculating the action integrals $\int_\gamma p dq$ along n independent cycles on those tori.

the complete solution of the system. However, recall that we are considering only the *formal* aspect. The problem of the *convergence* of the series is still open.

5 The problem of convergence

The question about the convergence of series containing small denominators is certainly the most challenging one in perturbation theory. Whittaker and Cherry based their hope that the series could be convergent on an example due to Bruns (see [49], chap. XVI, § 198). Whittaker was even more convinced in view of an example invented by himself: he could construct a two degrees of freedom Hamiltonian of the form (2.3) possessing a first integral independent of the Hamiltonian; the integral is given in closed form, and he checked that the first few terms of the expansion of the integral in power series do actually coincide with the expansion calculated with the method discussed in the previous section (see [49], chap. XVI, § 202). It was only in 1941 that Siegel [45] proved that the series so produced are generically non convergent.

Let me clarify the problem a bit more. Let me first show that the solution of a single eq. (4.5) for a first integral is harmless; the argument that follows is more general than Brun's example used by Whittaker and Cherry. If the known function $\Psi(p, q)$ is analytic, then the coefficients $c_k(p)$ of its Fourier expansion decay exponentially with the order $|k| = |k_1| + \dots + |k_n|$ of the Fourier mode, i.e., one has $|c_k(p)| \leq F e^{-|k|\sigma}$ for some positive constants F and σ . Assume now that the frequencies ω are constant, and recall the form (4.6) of the solution. It is a well known result of the Diophantine theory that the inequality

$$|\langle k, \omega \rangle| \geq \gamma |k|^{-\tau} \quad \text{for } 0 \neq k \in \mathbb{Z}^n \quad (5.1)$$

for some positive γ and some $\tau > n - 1$ is satisfied by a set of real vectors ω of large relative measure, the complement of this set having Lebesgue measure $O(\gamma)$.** Therefore the coefficients of the

**For $\tau = n - 1$ the set of real vectors ω satisfying (5.1) is non empty, but it has zero measure; for $\tau < n - 1$ it is empty in view of an approximation theorem by Dirichlet.

function $\Phi(p, q)$ in (4.6) are bounded by

$$\left| \frac{c_k(p)}{\langle k, \omega \rangle} \right| \leq \frac{F}{\gamma} |k|^\tau e^{-|k|\sigma},$$

i.e., they still decay exponentially. This assures that the Fourier series for $\Phi(p, q)$ is absolutely and uniformly convergent, so that $\Phi(p, q)$ is analytic. An argument of this type has been used by Poincaré ([42], tome II, chap. XIII, § 147). With a slightly more quantitative formulation it is one of the main tools for many recent results in perturbation theory, including, e.g., the proof of Kolmogorov's theorem.

However, the argument above is not sufficient in order to answer the question of convergence for the whole process of constructing a first integral. Indeed, the problem is that the system (4.3) must be solved *recursively*; hence at every step a new small denominator is added to the existing ones. The problem is that the *accumulation* of small denominators may cause the whole process to diverge. This is indeed what generically happens.^{††}

In order to illustrate the process of accumulation let me consider the simple case of the Hamiltonian

$$H(x, y) = \frac{1}{2} \sum_{l=1}^n \omega_l (x_l^2 + y_l^2) + H_3(x, y), \quad (5.2)$$

where $H_3(x, y)$ is a homogeneous polynomial of degree 3. Recall that the transformation to action-angle variables $x = \sqrt{2p} \cos q$, $y = \sqrt{2p} \sin q$ gives the Hamiltonian the form $H(p, q) = \sum_l \omega_l p_l + H_3(p^{1/2}, q)$. Remark also that a homogeneous polynomial of degree s in x, y is transformed to a trigonometric polynomial of degree s in q , and that solving the equation (4.3) for Φ_s when Ψ_s is a trigonometric polynomial of degree s generates only small denominators $\langle k, \omega \rangle$ with $|k| \leq s$; this is evident from the form (4.6) of the solution. Hence the worst possible denominator is

$$\alpha_s = \min_{0 < |k| \leq s} |\langle k, \omega \rangle|. \quad (5.3)$$

^{††}The accumulation of small denominators turns out to be quite fair if one uses a good algorithm for constructing a quasiperiodic solution as described by Lindstedt's series. This point is discussed, e.g., in [22].

The accumulation of small denominators is illustrated by the following table:

equation	degree	denominator
$\partial_\omega \Phi_3 = -\{H_3, \Phi_2\}$	3	α_3
$\partial_\omega \Phi_4 = -\{H_3, \Phi_3\}$	4	$\alpha_3 \alpha_4$
$\partial_\omega \Phi_5 = -\{H_3, \Phi_4\}$	5	$\alpha_3 \alpha_4 \alpha_5$
\vdots	\vdots	\vdots
$\partial_\omega \Phi_r = -\{H_3, \Phi_{r-1}\}$	r	$\alpha_3 \cdot \dots \cdot \alpha_r$

We choose $\Phi_2 = (x_l^2 + y_l^2)/2 = p_l$, one of the action variables. The first column is the equation for Φ_s , which is a homogeneous polynomial of degree s in (x, y) or a trigonometric polynomial in q in action-angle variables. By the way, we use the fact that the solution of the equation for Φ_s is a homogeneous polynomial of degree s ; this is easily checked. The third column gives the worst accumulation of small denominators: the Poisson bracket in the equation simply propagates the existing denominators, and the process of solving the equation adds the denominator of the corresponding order. Considering a neighbourhood Δ_ϱ of the origin of radius ϱ , we find for a generic term Φ_r of degree r an estimate

$$|\Phi_r(x, y)| \sim \frac{\varrho^r}{\alpha_3 \cdot \dots \cdot \alpha_r} \quad \text{for } (x, y) \in \Delta_\varrho.$$

If, according to the Diophantine estimate (5.3), we set $\alpha_s \sim \gamma |s|^{-\tau}$ then we get

$$|\Phi_r(x, y)| \sim \varrho^r (r!)^\tau. \quad (5.4)$$

Of course, this is not a proof that the series are not convergent: it is just an heuristic consideration suggesting that divergence should be the typical case (which has been proven by Siegel [45]).

The behaviour of the series may be investigated numerically by performing an explicit expansion with the help of a computer. Indeed, all functions involved in such a calculation are homogeneous polynomials, that may be easily represented in machine format by just storing the coefficients in an appropriate order. Moreover, all the process of construction of formal first integral reduces to simple algebraic manipulations of the coefficients of the polynomials (for a description of a program of this kind see, e.g., [15]).

Suppose that we have constructed a formal first integral

$$\Phi(x, y) = (x_2^2 + y_2^2)/2 + \Phi_3(x, y) + \cdots + \Phi_r(x, y)$$

up to some order r . Suppose for a moment that this is an exact first integral; then the orbit with initial point $(x^{(0)}, y^{(0)})$ must lie on the intersection of the surfaces $H(x, y) = H(x^{(0)}, y^{(0)})$ and $\Phi(x, y) = \Phi(x^{(0)}, y^{(0)})$, that is a two dimensional surface in the four dimensional phase space (remark that the functions H and Φ are clearly independent). The intersection of this surface with the plane $x_1 = 0$ gives a family of curves that may be projected in the plane (x_2, y_2) . The explicit construction of the curves is not difficult. Having fixed the value of the energy, say $H(x_1, x_2, y_1, y_2) = E$ we set $x_1 = 0$ and calculate $y_1 = \psi(x_2, y_2)$ by solving the equation $H(0, x_2, y_1, y_2) = E$. Then we just draw on the plane (x_2, y_2) the level lines of the function $\Phi(0, x_2, y_1, y_2) \big|_{y_1=\psi(x_2, y_2)}$.

We are now going to compare the Poincaré section obtained by numerical integration with the curves constructed using the first integrals. Having fixed the energy and the initial point $(x_2^{(0)}, y_2^{(0)})$ of an orbit we calculate on the one hand the Poincaré sections for that orbit and, on the other hand, the level lines $\Phi(0, x_2, y_1, y_2) \big|_{y_1=\psi(x_2, y_2)} = \Phi(0, x_2^{(0)}, y_1, y_2^{(0)}) \big|_{y_1=\psi(x_2^{(0)}, y_2^{(0)})}$. We expect that the points obtained by Poincaré section lie on the curves constructed via the first integral. Hence, a rough comparison may be made by simple inspection of the figures.

The results are presented in fig. 2. The energy value has been set to $H(x, y) = 0.0025$, and the first figure represents the Poincaré section for some orbits that surround the central periodic orbit. We forget the regions of chaotic orbits and of orbits that surround the $2/3$ resonance. The behaviour of the first integral is illustrated by drawing the level lines for different truncations of the series, from order 5 to order 70. One will notice that the best correspondence between Poincaré section and level lines is obtained by truncating the expansion of the first integral at order 9. Successive truncations to higher orders give raise to a sort of progressive destruction of the inner curves. Such a behaviour is reminiscent of that of asymptotic series. A more detailed phenomenological investigation of these phenomena may be found in [43] or [44].

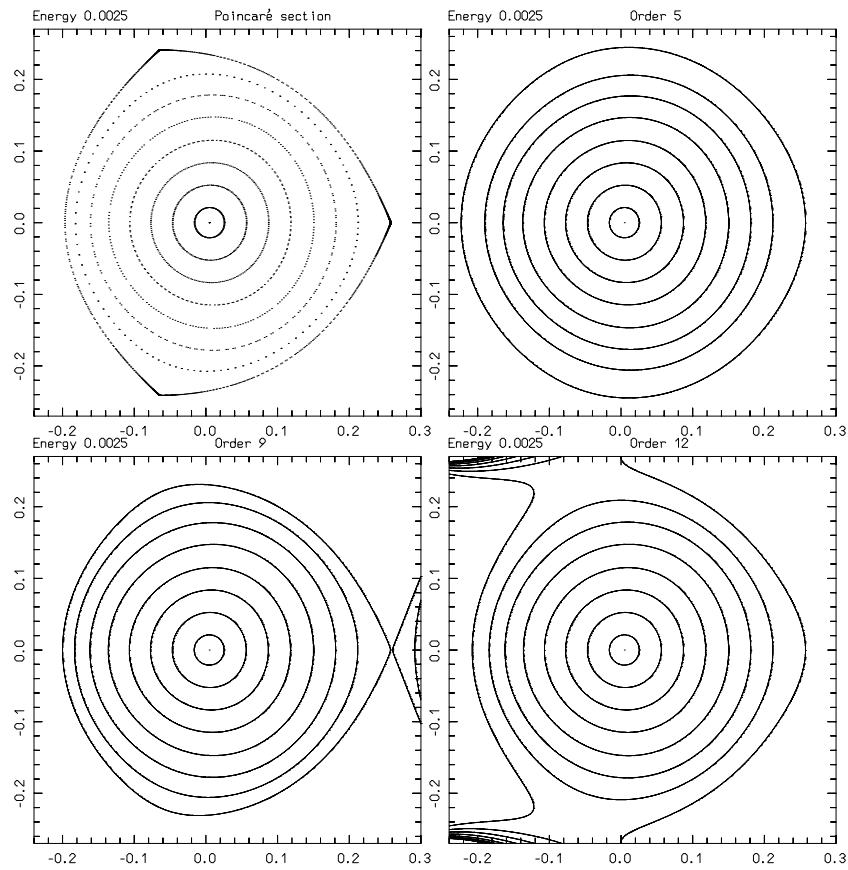


Figure 2: Comparison between the portrait of the Poincaré section and the level lines of a formal first integral. The first figure is the Poincaré section on the energy surface $H = 0.0025$. The remaining figures (see also the figure on next page) are the level lines of the formal first integral truncated at orders 5, 9, 12, 24, 33, 38, 45, 60 and 70. The levels drawn correspond to the values of the truncated integral at the initial points of the orbits represented in the Poincaré section.

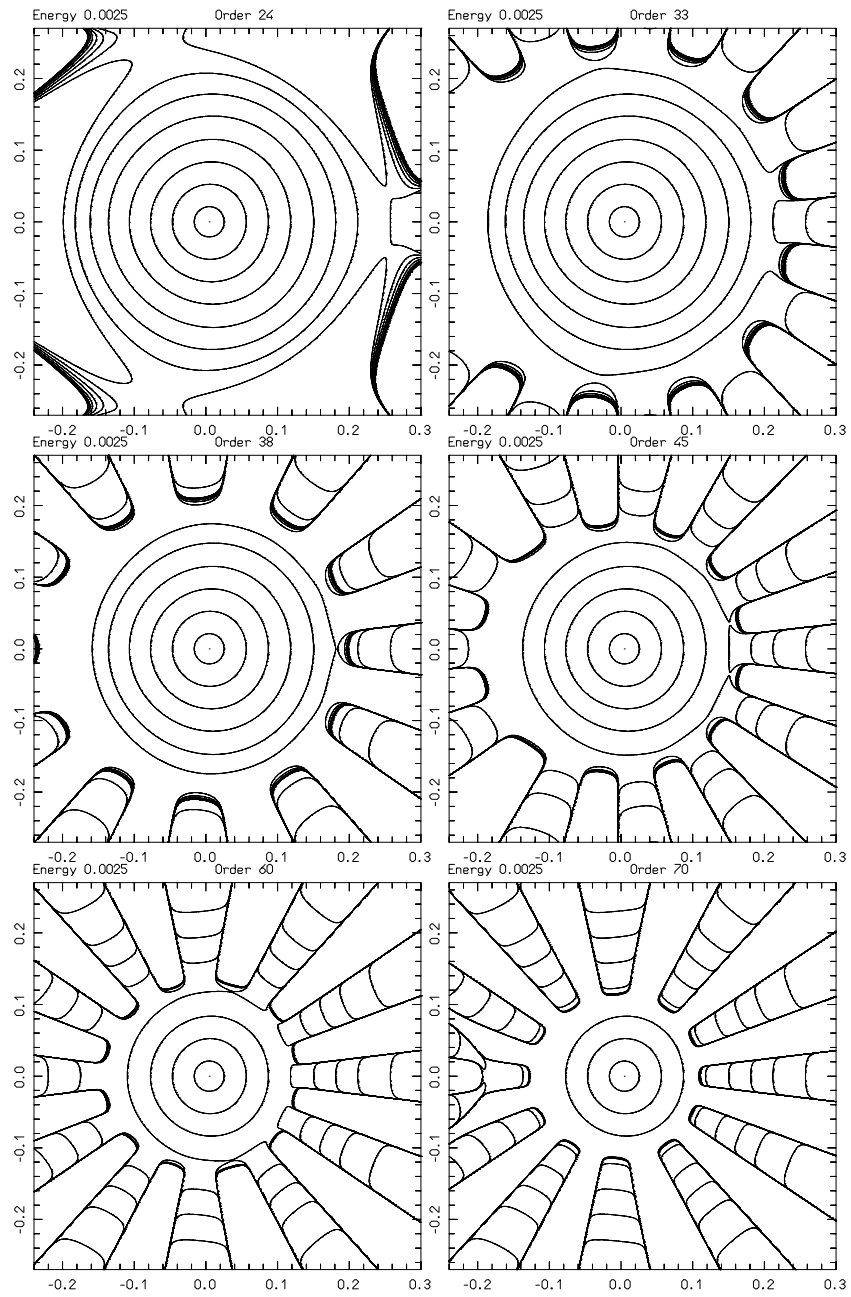


Figure 2: (continued.)

6 Exponential stability

In spite of their divergence, the formal first integrals constructed in the previous section are not useless. As a matter of fact, it is well known to astronomers since a long time that the series produced by perturbation methods are very useful in order to predict the planetary motions. We could even say that the big amount of work devoted by Poincaré to proving that divergence is the typical case for the series of perturbation theory has been essentially removed by the astronomers of this century. The best description of the situation, in my opinion, is still given by Poincaré in [42], tome II, chap. VIII. Having realized that the perturbation series present an asymptotic character, Poincaré's suggestion is to try to do the best use of the series so constructed. To this end he introduces the concept of formal expansion, namely a truncation of the perturbation expansions at a finite order. The exponential stability that I'm going to discuss here may be seen as a quantitative reformulation of Poincaré's program.

Following Poincaré, we simply truncate the construction of the formal integrals at a finite order r , thus constructing functions $\Phi(x, y)$ such that $\dot{\Phi} = \{H, \Phi\}$ is at least of degree $r + 1$ in (x, y) . Then we try to extract as much information as we can from those *truncated first integrals*.

Let me first give the heuristic argument of the previous section on the accumulation of small denominators a more precise formulation. This is just a (tedious) technical matter. Consider the polydisk with center at the origin and radius ϱ defined as

$$\Delta_\varrho = \left\{ (x, y) \in \mathbb{R}^n : x_l^2 + y_l^2 < \varrho^2, l = 1, \dots, n \right\}. \quad (6.1)$$

PROPOSITION 6.1 *Let the frequencies ω satisfy the Diophantine condition (5.3). Then there is a constant C such that the following holds true: for every $r > 2$ there exist n truncated first integrals*

$$\Phi^{(l,r)} = p_l + \Phi_3^{(l)} + \dots + \Phi_r^{(l)}, \quad p_l = \frac{x_l^2 + y_l^2}{2} \quad (6.2)$$

$(l = 1, \dots, n)$ such that for any $(x, y) \in \Delta_\varrho$ one has

$$|\dot{\Phi}^{(l,r)}| < C^r \varrho^{r+1} (r!)^{\tau+1}$$

For a proof see, e.g., [16].

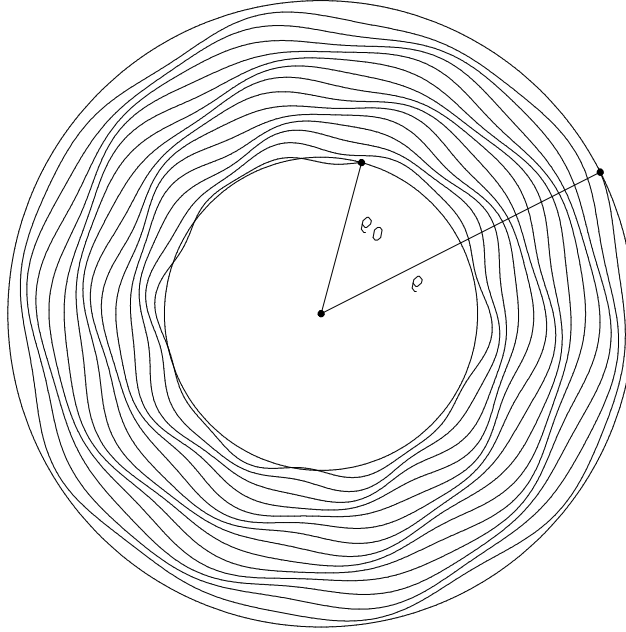


Figure 3: *Illustrating the concept of stability over a finite time. An orbit starting at a distance ϱ_0 from the center may evolve by steadily increasing its distance, until it eventually escapes the disk of radius ϱ .*

The problem of stability over a finite time may be formulated as follows (see fig. 3). Consider all orbits $(x(t), y(t))$ with initial point $(x(0), y(0)) \in \Delta_{\varrho_0}$ for some positive ϱ_0 . Choose $\varrho > \varrho_0$, e.g., let $\varrho = 2\varrho_0$, and prove that $(x(t), y(t)) \in \Delta_{\varrho}$ for $|t| \leq T(\varrho_0)$ with some “large” $T(\varrho_0)$, e.g., increasing to infinity as $\varrho_0 \rightarrow 0$. I will refer to $T(\varrho_0)$ as the *estimated stability time*. This seems to be a senseless question, since everybody who is familiar with the elementary theory of differential equation will immediately remark that this is just a property of continuity of the solutions with respect to initial data. The point, however, concerns the meaning of “large $T(\varrho_0)$ ”.

The request above may be meaningful if we take into consideration some characteristics of the dynamical system that is (more or less accurately) described by our equations. In this case “large” should be interpreted as *large with respect to some characteristic time of the physical system, or comparable with the lifetime of it*. For

instance, for the nowadays accelerators a characteristic time is the period of revolution of a particle of the beam and the typical lifetime of the beam during an experiment may be a few days, which may correspond to some 10^{10} revolutions; for the solar system the lifetime is the estimated age of the universe, which corresponds to some 10^{10} revolutions of Jupiter; for a galaxy, we should consider that the stars may perform a few hundred revolutions during a time as long as the age of the universe, which means that a galaxy does not really need to be much stable in order to exist.

From a mathematical viewpoint the word “large” is more difficult to explain, since there is no typical lifetime associated to a differential equation. Hence, in order to give the word “stability” a meaning in the sense above it is essential to consider the dependence of T on ϱ_0 . In this respect the continuity with respect to initial data does not help too much. For instance, if we consider the trivial example of the differential equation $\dot{x} = x$ one will immediately see that if $x(0) = x_0 > 0$ is the initial point, then we have $x(t) > 2x_0$ for $t > T = \ln 2$ no matter how small is x_0 ; hence T may hardly be considered to be “large”, since it remains constant as x_0 decreases to 0. Conversely, if for a particular system we could prove, e.g., that $T(\varrho_0) = O(1/\varrho_0)$ then our result would perhaps be meaningful; this is indeed the typical goal of the theory of adiabatic invariants.

Coming back to our problem we may proceed as follows. The condition $(x, y) \in \Delta_\varrho$ is equivalent to $p_l < \varrho^2/2$ for $l = 1, \dots, n$ (remark that $p_l \geq 0$ by definition). Let us now use the elementary inequality

$$\begin{aligned} |p(t) - p(0)| &\leq |p(t) - \Phi^{(r,l)}(t)| + |\Phi^{(r,l)}(t) - \Phi^{(r,l)}(0)| + \\ &\quad + |\Phi^{(r,l)}(0) - p(0)|. \end{aligned} \quad (6.3)$$

Let me define

$$\delta_r(\varrho) = \max_l \sup_{(x,y) \in \Delta_\varrho} |\Phi^{(r,l)}(x, y) - p_l(x, y)|; \quad (6.4)$$

then we have $(x(t), y(t)) \in \Delta_\varrho$ provided

$$|\Phi^{(l,r)}(t) - \Phi^{(l,r)}(0)| < D_r(\varrho_0, \varrho) := \frac{\varrho^2 - \varrho_0^2}{2} - \delta_r(\varrho_0) - \delta_r(\varrho). \quad (6.5)$$

In view of (6.2) we have $\delta_r(\varrho) = O(\varrho^3)$. On the other hand, by the theorem above we have $|\dot{\Phi}^{(r,l)}| < B_r \varrho^{r+1}$, with some constant B_r . Let us choose $\varrho > \varrho_0$ such that $D_r(\varrho_0, \varrho) = O(\varrho^3)$; then we conclude that $(x(t), y(t)) \in \Delta_\varrho$ for $|t| < T(\varrho_0) = O(1/\varrho_0^{r-2})$. For instance, for $r = 3$ we have $T(\varrho_0) = O(1/\varrho_0)$, namely the typical estimate of the theory of the adiabatic invariants. The more general estimate for an arbitrary r was used by Birkhoff as a basis for his theory of *complete stability* (see [4], chap. IV, § 2 and § 4). By the way, Birkhoff could not do better because he did not try to evaluate the constant B_r .^{‡‡}

The exponential stability follows by exploiting the asymptotic character of the series representing the first integrals. Let ϱ be fixed and small enough, and recall again that, by the theorem above, we have $|\dot{\Phi}^{(r,l)}| = O(\varrho^{r+1}(r!)^{\tau+1})$. If we let r to increase the quantity $\varrho^r (r!)^a$ will first decrease until $r \leq 1/\varrho^{1/a}$ and then it will start to increase. Therefore we just stop when our estimate has reached the minimum. This means that we determine an optimal value r_{opt} of r as a function of ϱ , namely $r_{\text{opt}} = 1/\varrho^{1/a}$. The exponential estimate follows by a straightforward use of Stirling's formula, since

$$\varrho^{r_{\text{opt}}(r_{\text{opt}}!)^a} \sim \varrho^{r_{\text{opt}}} \left(\frac{r_{\text{opt}}}{e}\right)^{ar_{\text{opt}}} = \exp \left[-a \left(\frac{1}{\varrho}\right)^{1/a} \right].$$

A more precise formulation is the following

THEOREM 6.1 *Consider the Hamiltonian (2.3), and assume that the frequencies ω satisfy the Diophantine condition (5.1). Then there exist positive constants A, C and ϱ_* such that for every $\varrho < \varrho_*$ the following holds true: there are n independent functions $\Phi^{(1)}, \dots, \Phi^{(n)}$*

^{‡‡}As we have seen, the estimate $B_r = O(r!)$ follows from the hypothesis that the frequencies satisfy a Diophantine condition (5.3). The relevance of conditions of this kind for problems with small denominators was first pointed out in 1942 by Siegel [46] in connection with the problem of convergence of the so called *Schröder series for the center problem*. After Siegel, conditions of Diophantine type have become a standard tool of KAM theory.

such that for every $(x, y) \in \Delta_\varrho$ we have

$$\begin{aligned} |\Phi^{(l)}(x, y) - p_l(x, y)| &< C \left(\frac{\varrho}{\varrho_*} \right)^3, \\ |\dot{\Phi}^{(l)}(x, y)| &\leq A \exp \left[-(\tau + 1) \left(\frac{\varrho_*}{\varrho} \right)^{1/(\tau+1)} \right]. \end{aligned}$$

For a proof see [16]. The exponential estimate for the stability time $T(\varrho_0)$ follows by repeating almost word by word the argument above leading to $T(\varrho_0) = O(1/\varrho_0^{r-2})$.*

I emphasize that the good choice of the order r of truncation of the series is the key of all the results of exponential stability in the light of Nekhoroshev's theory. The present discussion concerns only the case of the equilibrium point, which is the simplest one. Nekhoroshev's theorem however applies to the more general system (2.6) with some extra conditions on the unperturbed Hamiltonian $h(p)$. For complete proofs of Nekhoroshev's theorem see, e.g., [39, 40, 2, 3, 32] and [18]. For a general discussion of the problem of stability in Hamiltonian systems see [23]. For a stronger stability result and the relations between Nekhoroshev's theory and KAM theory see [33, 34] and [20].

7 A numerical application

The application to physical systems of the results above on exponential stability is not straightforward. As stated by the theorem, there is a threshold ϱ_* above which the theory may not be applied; on the

*A comparison with Littlewood's estimate reported at the beginning of sect. 1 shows that we get an exponential law of the form $\exp(\varrho^{-1/(\tau+1)})$, while Littlewood finds $\exp(\varrho^{-1/2} |\ln \varrho|^{-3/4})$ (recall that ϱ is the actual perturbation parameter, named ε by Littlewood). Our exponent $1/(\tau + 1)$ in place of Littlewood's $1/2$ is due to the higher generality of our estimate, which applies to any finite number n of degrees of freedom. In the best case we may set $\tau = n - 1$, which gives an exponent $1/n$; the case investigated by Littlewood corresponds to $n = 2$, so that agreement is found. The factor $|\ln \varrho|^{-3/2}$ is found by Littlewood because he does not use the Diophantine inequality. His argument is based on a property of the continuous fraction representation of the ratio between the frequencies. However, his method may hardly be extended to more than 2 frequencies (i.e., to $n > 2$).

other hand, the existence of a threshold is typical of all results of perturbation theory. Now, the problem is whether or not the perturbation acting on a real physical system is smaller than the threshold.

The answer to such a question is not trivial. When the KAM theorem was established it was quite common to believe that this was the proof that the solar system is stable in probabilistic sense, since the set of initial data leading to quasiperiodic motions on invariant tori has relative measure close to one. But the best analytical estimates available at that time could only prove, rigorously speaking, that KAM stability is assured provided the mass of Jupiter is less than 10^8 times the mass of a proton. Things were even worse for Nekhoroshev's theorem; for, according to the available estimates, the mass of Jupiter should be several order of magnitude less than the mass of a proton. Of course, everybody was well aware that the analytical estimates are very crude: trusting the applicability of KAM theory to the solar system was just matter of being optimist.

Recent developments of our knowledge on this matter have shown that things are much more complicated than it was expected — as usual. According to a numerical integration of the orbits of all planets over 10^{10} years made by Laskar [26], only the motion of the major planets (Jupiter, Saturn, Uranus and Neptune) may be confidently considered to be quasiperiodic on an invariant torus. The orbits of all minor planets (including the earth) present instead a significant chaotic component. The role of resonances in producing stable states or chaotic dynamics is even more evident if one considers the orbits of the asteroids. According to Guzzo and Morbidelli interpreting such a complicated dynamics in terms of Nekhoroshev's theory is a major challenge [35].

A possible method for improving the estimates of the thresholds for applicability of our theories on exponential stability is to perform explicitly the series expansions required by perturbation theory. It is not recommended to do it by hand, of course. However, it should be remarked that most of perturbation theory relies on simple algebraic operations that may be effectively programmed on computers. The development of packages of algebraic manipulation especially devoted to the needs of perturbation theory has been started quite soon. Concerning the particular problem of first integrals for the

case of the elliptic equilibrium the first program – up to my knowledge – has been implemented by Contopoulos around 1960.

In order to check the effectiveness of exponential stability in a simple but interesting case let me consider the triangular Lagrangian equilibria of the restricted problem of three bodies in the planar case, i.e., the Littlewood's problem mentioned at the beginning of this note. The goal is to evaluate the size of the region of stability in the case Sun–Jupiter. This case is particularly interesting for two reasons. Firstly, it is simple enough to allow us to expand the perturbations series up to a reasonably high order, thanks to the computational power of the computers available nowadays.[†] Secondly, in the neighbourhood of the Lagrangian points there are several asteroids, called Trojan, that have been observed, so that a comparison of the theoretical results with reality is possible – inasmuch the planar restricted problem of three bodies may be considered as an appropriate model of the real world.

The construction of formal integrals may be performed via either method, constructing the direct expansion of the series with the algorithm discussed in sect. 5 or going through the process of constructing the normal form of Birkhoff (that I just mentioned above, without entering the details). This requires a preliminary expansion of the Hamiltonian in the neighbourhood of the equilibrium point, and a transformation to coordinates that give the quadratic part of the Hamiltonian a diagonal form $\sum_l \omega_l (x_l^2 + y_l^2)/2$. In the case of the triangular Lagrangian point L_4 for the Sun–Jupiter system the frequencies turn out to be $\omega_1 \sim 0.99676$ and $\omega_2 \sim -0.80464 \times 10^{-1}$.[‡]

[†]In order to give a concrete idea of the advantages offered by the nowadays computers, let me mention that at the beginning of the sixties Contopoulos computed the formal expansions of first integrals up to order 6 or 7; in 1966 Gustavson could reach the order 8. In 1978, when I implemented the small package used in the calculations of the present paper, I could reach the order 15 on a CDC 7600, the most powerful (and expensive) computer available at that time. The calculation of the series up to order 70 that has produced the results illustrated in sect. 5 has been made on a PC computer with a Pentium 200 processor and with 64 Mbytes of RAM. It should be remarked that the most severe limitation in this kind of calculation is due to memory, since the number of coefficients to be stored grows quite fast with the order.

[‡]For the Hamiltonian (2.3) the problem of stability is a simple one in case the frequencies have all the same sign, e.g., they are all positive. For, in this case

In the present calculation the construction of Birkhoff's normal form up to a finite arbitrary order r has been used; this gives some advantages that I briefly illustrate.

The procedure consists in constructing a near to identity canonical transformation as a polynomial of degree $r - 1$, e.g.,

$$\begin{aligned} x &= x' + \varphi_2(x', y') + \cdots + \varphi_{r-1}(x', y') \\ y &= y' + \psi_2(x', y') + \cdots + \psi_{r-1}(x', y'), \end{aligned}$$

such that the transformed Hamiltonian is given the form, called Birkhoff's normal form,

$$\begin{aligned} H'(x', y') &= \sum_l \omega_l p'_l + Z^{(r)}(p'_l) + \mathcal{R}^{(r)}(x', y'), \\ p'_l &= \frac{1}{2}(x_l'^2 + y_l'^2), \end{aligned} \tag{7.1}$$

where $Z^{(r)}(p')$ is a (non homogeneous) polynomial of degree r in (x', y') starting with terms of degree 4 that is actually a function only of the new action variables p' , and $\mathcal{R}^{(r)}(x', y')$ is a remainder that is still unnormalized, and is in fact a power series starting with terms of degree $r + 1$ in (x, y) . The new action variables p'_1, \dots, p'_n are the truncated first integrals[§] that we are going to use. Remark that we have $\dot{p}' = \{H, p'\} = \{\mathcal{R}^{(r)}, p'\}$.

the Hamiltonian has a minimum at the equilibrium, and so it may be used as a Lyapounov function. However, this theory may not be used if the frequencies have different signs, as in our case. For this reason the problem of stability of the triangular Lagrangian points is essentially still open. Full conclusions may be drawn only in the planar case of two degrees of freedom using the KAM theory. However, the argument may not be extended to the spatial case with three degrees of freedom. Moreover, there are no explicit estimates of the size of the region where KAM theory may be applied. In a sense, the present study of stability over finite but large times is useless, since it is limited to the planar case. But I emphasize that there is no difficulty in extending the same method to the spatial case or even to take into account the ellipticity of Jupiter's orbit. The choice of studying the planar case was just dictated by the possibility of pushing the expansions to a higher order, which may give a more precise idea of the limits of the method. For a study in the spatial case see, e.g., [47] or [6].

[§]The first integrals obtained with the direct construction of sect. 5 do not coincide with the functions p'_1, \dots, p'_n obtained via the transformation to Birkhoff's normal form, but are functions of them.

All quantities mentioned here may be explicitly constructed by algebraic manipulation up to some (not too low) order. Therefore the analysis of stability may be performed in the new variables (x', y') making reference to the Hamiltonian (7.1); this simplifies a lot the estimates of stability, because we avoid evaluating the difference $|p' - p|$ required by (6.3) and (6.4). Moreover, we may forget the exponential law for the stability time, and use our complete knowledge of all functions in order to perform an optimization “by hand” of our estimates.

Let the domain Δ_{ϱ_0} of the initial data be defined as a polydisk in the new variables (x', y') . By the same argument used in sect. 6 we know that an orbit with initial point in Δ_{ϱ_0} can not escape from Δ_{ϱ} for $|t| < \tau(\varrho_0, \varrho, r)$, where

$$\tau(\varrho_0, \varrho, r) = \frac{\varrho^2 - \varrho_0^2}{2F(\varrho, r)}, \quad F(\varrho, r) = \max_l \sup_{(x', y') \in \Delta_{\varrho}} |\{\mathcal{R}^{(r)}, p'_l\}|. \quad (7.2)$$

The quantity $F(\varrho, r)$ may be evaluated by using the first term of the expansion of the remainder $\mathcal{R}^{(r)}$, that in turn may be explicitly calculated. This produces an estimate depending on the arbitrary quantities ϱ and r .

Let now ϱ_0 and r be fixed; then, in view of $F(\varrho, r) = O(\varrho^{r+1})$, the function $\tau(\varrho_0, \varrho, r)$, considered as function of ϱ only, has a maximum for some value ϱ_r . This looks quite odd, because one would expect τ to be an increasing function of ϱ . However, recall that (7.2) is just an estimate; looking for the maximum means only that we are trying to do the best use of our poor estimate. Let us now keep ϱ_0 constant, and calculate $\tau(\varrho_0, \varrho_r, r)$ for increasing values of $r = 1, 2, \dots$, with ϱ_r as above. Since $F(\varrho, r)$ is expected to grow quite fast with r we expect to find a maximum of $\tau(\varrho_0, \varrho_r, r)$ for some optimal value r_{opt} . Thus, we are authorized to conclude that *for every ϱ_0 we can explicitly evaluate the positive constants $\varrho(\varrho_0) = \varrho_{r_{\text{opt}}}$ and $T(\varrho_0) = \tau(\varrho_0, \varrho(\varrho_0), r_{\text{opt}})$ such that an orbit with initial point in the polydisk Δ_{ϱ_0} will not escape from Δ_{ϱ} for $|t| < T(\varrho_0)$.*

Let me summarize the results obtained for the Lagrangian point L_4 in the Sun–Jupiter case (for a full discussion see [19]). All series have been computed up to order $r_{\text{max}} = 34$. The graph of the optimal

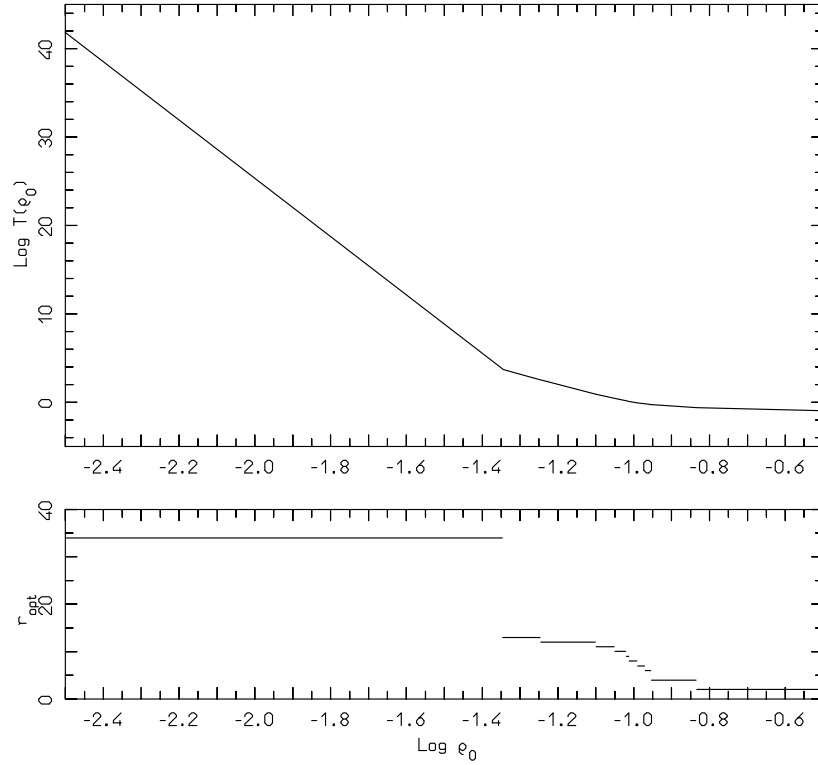


Figure 4: *Upper figure: estimated stability time as a function of the size ϱ_0 of the domain containing the initial data. Lower figure: the optimal order r_{opt} as a function of ϱ_0 .*

order r_{opt} and of the estimated stability time T as functions of ϱ_0 are reported in fig. 4. The time unit is the period of revolution of Jupiter divided by 2π ; the estimated age of the universe is about 10^{10} time units. I emphasize that a small change of ϱ_0 may change significantly the estimate of the stability time. Moreover, it should be stressed that in our calculation the optimal order may not exceed the value $r_{\text{max}} = 34$; thus the estimates could be further improved, in principle. The value ϱ_0 for which the estimated stability time is the age of the universe corresponds roughly to 0.127 times the distance L_4 -Jupiter. Hence the result is clearly realistic.

Table 1: *Estimated stability region for the known asteroids. The first column gives the catalog number. The second column gives the value of ϱ which ensures stability over the age of the universe; the asteroid is inside if $\varrho > 1$ (see text). The table is sorted in decreasing order with respect to the stability parameter ϱ .*

88181612	1.487790	4827	2.868400×10^{-1}
89211605	1.135130	4722	2.755600×10^{-1}
41790004	1.100990	1173	2.721800×10^{-1}
1870	1.048060	10240002	2.434500×10^{-1}
2357	8.470200×10^{-1}	2594	2.360100×10^{-1}
5257	7.504500×10^{-1}	4829	2.358500×10^{-1}
88181912	6.597200×10^{-1}	88180812	2.247200×10^{-1}
5233	6.495000×10^{-1}	4754	2.157600×10^{-1}
4708	6.275300×10^{-1}	4707	2.138800×10^{-1}
88181311	6.063800×10^{-1}	43170004	2.106900×10^{-1}
1871	6.000700×10^{-1}	89210305	2.032200×10^{-1}
31080004	5.956600×10^{-1}	88182012	1.989500×10^{-1}
94031908	5.928600×10^{-1}	4805	1.974600×10^{-1}
2674	5.894200×10^{-1}	5511	1.908600×10^{-1}
88180412	5.876200×10^{-1}	89211505	1.890100×10^{-1}
88180710	5.425600×10^{-1}	20350004	1.838900×10^{-1}
88191102	4.979700×10^{-1}	884	1.820300×10^{-1}
88182510	4.658500×10^{-1}	2893	1.758800×10^{-1}
2207	4.487900×10^{-1}	1872	1.723100×10^{-1}
89201902	4.163900×10^{-1}	88181213	1.673900×10^{-1}
94031500	4.075300×10^{-1}	51910004	1.644500×10^{-1}
89212405	4.005000×10^{-1}	4828	1.637500×10^{-1}
89211705	3.826400×10^{-1}	5130	1.629200×10^{-1}
5907	3.790100×10^{-1}	5476	1.483600×10^{-1}
88181411	3.757900×10^{-1}	88181410	1.362000×10^{-1}
4792	3.617700×10^{-1}	88191602	1.333500×10^{-1}
88180811	3.519900×10^{-1}	2223	1.278500×10^{-1}
3240	3.359200×10^{-1}	40350004	1.272900×10^{-1}
5638	3.162000×10^{-1}	88180813	1.255000×10^{-1}
43690004	3.061600×10^{-1}	2241	1.239700×10^{-1}
31630002	3.046700×10^{-1}	6002	1.230000×10^{-1}
4348	2.977800×10^{-1}	88192301	1.214500×10^{-1}

3708	1.171100×10^{-1}	88181510	7.132770×10^{-2}
88190103	1.162400×10^{-1}	88182511	6.839150×10^{-2}
88181810	1.144900×10^{-1}	5648	6.776230×10^{-2}
87171400	1.106500×10^{-1}	88191203	6.354580×10^{-2}
88191003	1.089100×10^{-1}	5637	6.081990×10^{-2}
5119	1.086700×10^{-1}	90202212	5.963150×10^{-2}
88180512	1.079300×10^{-1}	2895	5.746530×10^{-2}
88190703	1.078800×10^{-1}	5120	5.713580×10^{-2}
1172	1.046900×10^{-1}	3451	5.705220×10^{-2}
31040004	9.614220×10^{-2}	4791	5.332690×10^{-2}
4715	9.453910×10^{-2}	4709	5.294080×10^{-2}
4832	9.399910×10^{-2}	3317	4.989590×10^{-2}
90221206	8.377690×10^{-2}	4867	4.901550×10^{-2}
1873	8.205510×10^{-2}	1867	4.773260×10^{-2}
88180701	7.394110×10^{-2}	88172500	4.017310×10^{-2}
41010004	7.333020×10^{-2}	1208	3.597040×10^{-2}
617	7.324830×10^{-2}	2363	3.573360×10^{-2}

Now, it is natural to ask if the theory above may be effectively applied to the Trojan asteroids that are known to exist in the vicinity of the triangular Lagrangian points of the Sun-Jupiter system. That is, if their stability may be assured, if not forever, at least for a sufficiently long time.

I have to say that the application of the method above to the existing asteroids is a bit disappointing. Using the data for from Marsden's catalog of 1990 we investigate if some asteroid is inside a region that assures stability for the age of the universe. The results are summarized in table 1. The catalog contains 98 asteroids that are in the region of libration around the Lagrangian point L_4 . The orbital elements of the asteroid at a given epoch have been used as initial datum; having performed all necessary transformations the distance (in phase space) of the initial point from the equilibrium L_4 has been calculated in the coordinates of the normal form; let us denote that distance by $\bar{\varrho}$. Then we imagine that we can move the initial datum along the line (still in phase space) joining its current position to the equilibrium; acting so, we calculate the distance ϱ_0 such that stability is assured for a time as long as the age of the universe. The ratio $\varrho_0/\bar{\varrho}$ is reported in the second column of the

table; thus, a value bigger than 1 means that the asteroid is inside the estimated stability region.

It is seen that (only) 4 asteroids are inside. About 40 percent of the asteroids could be taken inside the stability region if we could improve our estimates by a factor 5, that is not too big. The worst case requires a factor 30. However, it should be remarked that improving the estimates by a factor bigger than 8 will be impossible, because Jupiter would fall inside the region of stability. This apparent nonsense is mainly due to the fact that the asteroids can not get too close to Jupiter, but can move quite far from the point L_4 in the direction opposite to Jupiter. Indeed, a numerical exploration shows that the orbits of the stable asteroids fill a banana-shaped region that extends along the circle passing through the point L_4 and having its center on the Sun. The asteroid may eventually reach an angular position that is very far from Jupiter. On the other hand, the coordinates used in the calculation described here are symmetric. This causes a significant cut off that excludes many asteroids. It is reasonable to expect that introducing suitable coordinates that take into account the lack of symmetry of the true region of stability would significantly improve the result.

8 Conclusions

The theory of exponential stability initiated by Moser and Littlewood more than 40 years ago, and fully stated by Nekhoroshev, appears as the most natural outcome of a careful analysis of the asymptotic behaviour of the series produced by perturbation theory. It also appears as the only method available for proving the stability of a realistic physical system — like the solar system or some parts of it — for a set of initial conditions compatible with our experimental knowledge of the initial data and of the parameters of the system. The problem of investigating if the phenomenon of exponential stability is effective for a real system, where the perturbations are not arbitrarily small, is still open, and is a challenging one. However, at least in the simple case of the Trojan asteroids we have found that stability for the age of the solar system is likely to occur.

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