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# WHAT WE KNOW ABOUT THE SECOND ADJUNCTION MAPPING

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ABSTRACT. Let $\hat{X}$ be an algebraic submanifold of complex projective space $\mathbb{P}^N$. Assume that $n := \dim \hat{X} \geq 3$ and let $\hat{L}$ denote the restriction of the hyperplane section bundle $\mathcal{O}_{\mathbb{P}^N}(1)$ to $\hat{X}$. The meromorphic map $\hat{\Phi}_k$ associated to $|k(K_{\hat{X}} + (n-2)\hat{L})|$ for $k \geq 1$ ties together the pluricanonical maps of the surface sections of $\hat{X}$. Known results show that the behavior of $\hat{\Phi}_k$ is far better than one would expect from experience with the pluricanonical mappings of algebraic surfaces. In this article we discuss the known results on the structure of the mappings $\hat{\Phi}_k$ and describe the open problems.

## Introduction

Let $\hat{X}$ be an algebraic submanifold of complex projective space $\mathbb{P}^N$. Assume that $n := \dim \hat{X} \geq 3$ and let $\hat{L}$ denote the restriction of the hyperplane section bundle $\mathcal{O}_{\mathbb{P}^N}(1)$ to $\hat{X}$. The meromorphic map $\hat{\Phi}_k$ associated to $|k(K_{\hat{X}} + (n-2)\hat{L})|$ for $k \geq 1$ ties together the pluricanonical maps of the surface sections of $(\hat{X}, \hat{L})$. Known results show that the behavior of $\hat{\Phi}_k$ is far better than one would expect from experience with the pluricanonical mappings of algebraic surfaces. As we will see below, questions about the $\hat{\Phi}_k$ quickly reduce to questions about slightly better behaved maps $\Phi_k$: the map $\hat{\Phi}_k$ is in the same relation to the map $\Phi_k$ as the map associated to $|kK_{\hat{S}}|$ on a surface $\hat{S}$ with a minimal model $S$ is to the map associated to $|kK_S|$.

M. Beltrametti and this author have over the last few years [13, 14, 15, 16, 17] carried out an investigation of the maps $\hat{\Phi}_k$. Though many basic questions remain open, real progress has been made. Since most of this work has not yet appeared and since what is known is scattered over the literature, it is a very good time to survey what is known about the second adjunction mapping $\Phi_1$ and the related mappings $\Phi_k$ for $k \geq 2$. The question of when the second adjunction mapping $\Phi_1$ exists as a morphism and not only as a meromorphic map is one of the two main open problems in adjunction theory (the other problem is about spectral values [18, Conjecture A.5 of the Appendix]).

For most of the results that we discuss, the real work takes place in proving the three dimensional case. The general situation with $n := \dim \hat{X} \geq 3$ usually follows from the case of $n = 3$ by a straightforward induction. Except in the last section of this survey, the reader will not lose much by setting the dimension $n$ to 3. I have included a few new results, e.g., Theorems 2.2, 2.4, 3.2, 3.5, 3.7, 3.9, and 3.10; and the Corollaries 2.3 and 3.8 that arose in the process of writing this survey. Background material and notation are presented in §1. In §1 we also explain the general problem this article is about and explain how the problem naturally breaks up into a number of smaller problems. In §2 we discuss known spannedness results for the line bundles that arise in the study of the maps $\hat{\Phi}_k$. In §3 we discuss $\hat{\Phi}_k$ when $\kappa(K_{\hat{X}} + (n-2)\hat{L}) \leq 2$. In §4 we discuss $\hat{\Phi}_k$ when $\kappa(K_{\hat{X}} + (n-2)\hat{L}) \geq 3$ and $\dim \hat{\Phi}_1(\hat{X}) \leq 3$. In §5 we discuss what we know about $\hat{\Phi}_2$ under the condition that $\kappa(K_{\hat{X}} + (n-2)\hat{L}) \geq 3$. Throughout the paper we have listed open problems. I would like to express my deep thanks to the *Rendiconti del Seminario Matematico e Fisico di Milano*, the University of Milan, and especially to Professor Antonio Lanteri and Professor Marino Palleschi for giving me this opportunity to put my thoughts on this topic in writing. I would also like to thank Professor Beltrametti and Professor Lanteri for their very helpful comments about this article.

## Contents

## 1 Background material and the refinement of the basic problem

Throughout this paper we deal with meromorphic maps $\gamma : V \to W$ between irreducible projective varieties. By definition, a *meromorphic map* $\gamma : V \to W$ between irreducible projective varieties is an irreducible subvariety of $\Gamma \subset V \times W$ whose projection onto $V$ under the product projection $V \times W \to V$ is generically one-to-one. $\Gamma$ is called the *graph* of $\gamma$. If $V$ is normal, then $\gamma$ defines (i.e., $\Gamma$ is the graph of a well defined) morphism off of a set of codimension at least two in $V$. We say that $\gamma$ is surjective (respectively, birational) if the image of $\Gamma$ to $W$ under the product projection $V \times W \to W$ is onto (respectively, generically one-to-one). Given a surjective meromorphic mapping $\gamma : V \to W$ between irreducible projective varieties, a general fiber of $\gamma$ is defined as the image in $V$ under the product projection of a general fiber of $\Gamma \to W$.

**Example 1.1** The following is the typical sort of meromorphic map that occurs in this article. Let $\mathcal{K}$ denote a line bundle on a projective variety $V$. If $h^0(\mathcal{K}) \geq 1$ we obtain a morphism $V - B \to \mathbb{P}^{h^0(\mathcal{K})-1}$ on the complement of the base locus $B$ of $|\mathcal{K}|$. We let $\Gamma$ denote the closure of the graph of this morphism in $V \times \mathbb{P}^{h^0(\mathcal{K})-1}$.

Many of the usual operations with morphisms extend to meromorphic maps. For example if $\alpha : A \to B$ and $\beta : B \to C$ are surjective meromorphic maps with graphs $\Gamma_\alpha$ and $\Gamma_\beta$, then the composition $\beta \circ \alpha : A \to C$ is defined by the image $\Gamma_{\beta\circ\alpha}$ of $\Gamma_\alpha \times \Gamma_\beta$ in $A \times C$ under the product projection $A \times B \times B \times C \to A \times C$. It is straightforward to check that in the case $\alpha, \beta$ are morphisms, then $\Gamma_{\beta\circ\alpha}$ agrees with the graph of the composition $\beta \circ \alpha$. Similarly the Remmert-Stein factorization of a meromorphic map $\gamma : V \to W$ between irreducible projective varieties is easily defined. For simplicity we assume that $V$ is normal. Let $\nu : \overline{\Gamma} \to \Gamma$ denote the normalization morphism of the graph $\Gamma$ of $\gamma$. We let $G : \overline{\Gamma} \to W$ denote the composition of $\nu$ with the restriction of the product projection to $\Gamma$. Thus we can Remmert-Stein factorize $G = s \circ \overline{r}$ as the composition of a morphism $\overline{r} : \overline{\Gamma} \to Z$ with connected fibers onto a normal variety $Z$ followed by a finite-to-one morphism $s$. We can Remmert-Stein factorize $\gamma = s \circ r$, where $r$ is the meromorphic map $r : V \to Z$ whose graph is the image of the graph of $\overline{r}$ in $V \times Z$. Note that the general fiber of $r$ is irreducible and $s$ is a finite morphism. When dealing with a line bundle or vector bundle $\mathcal{V}$ on an algebraic set $V$, we often denote the restriction of $\mathcal{V}$ to an algebraic subset $A$ of $V$ by $\mathcal{V}_A$. Similarly given a map $g : V \to W$ between algebraic sets we often denote the restriction of $g$ to an algebraic subset $A$ of $V$ by $g_A$. Let $\widehat{L}$ be a very ample line bundle on a complex connected $n$-dimensional projective manifold $\widehat{X}$. By a *curve section* (respectively, a *surface section*; respectively, a *t-fold section*) of $(\widehat{X}, \widehat{L})$, we mean a curve $C \subset \widehat{X}$ (respectively, a surface $\widehat{S} \subset \widehat{X}$;

respectively, a $t$-dimensional algebraic subset $Z \subset \widehat{X}$) obtained as the intersection of $\widehat{X}$ (embedded in projective space $\mathbb{P}^N$ by $|\widehat{L}|$) with a codimension $n-1$ linear subspace of $\mathbb{P}^N$ (respectively, a codimension $n-2$ linear subspace of $\mathbb{P}^N$; respectively, a codimension $n-t$ linear subspace of $\mathbb{P}^N$). More generally, given a line bundle $\mathcal{L}$ on an irreducible $n$-dimensional variety $V$, by a *$t$-fold section* of $(V, \mathcal{L})$ we will mean a $t$-dimensional algebraic subset $Z \subset V$ which is equal to the intersection $\cap_{i=1}^{n-t} H_i \subset V$ of $n-t$ elements $H_1, \dots, H_{n-t} \in |\mathcal{L}|$. We will use *curve section* (*surface section*) of $(V, L)$ for 1-fold sections (2-fold sections) of $(V, L)$. We say that a line bundle $\mathcal{L}$ on a variety $V$ is *spanned* if, for each point $x \in V$, there is a global section of $\mathcal{L}$ that does not vanish at $x$. We need some results about the first adjunction mapping. Because they are a model for the type of result we would like to eventually have for the second adjunction mapping, we present them in some detail. We compromise here in that we restrict ourselves to the situation when $n \geq 3$, even though for the first adjunction mapping the surface case is the hardest case (with a number of exceptional polarized surfaces with no higher dimensional analogues). I refer to my book [12] with Beltrametti for a detailed discussion with the full history of classical adjunction theory and its generalizations. Besides [12], I refer to Sommese [31, 32] and Van de Ven [40] for the following result.

**Theorem 1.2** *Let $\widehat{L}$ be a very ample line bundle on a complex connected $n$-dimensional projective manifold $\widehat{X}$ of dimension $n \geq 3$. Then either $K_{\widehat{X}} + (n-1)\widehat{L}$ is spanned, or $(\widehat{X}, \widehat{L})$ is one of the following:*

1. *$(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$; or*
2. *a quadric in $\mathbb{P}^{n+1}$, i.e., $\widehat{X} \in |\mathcal{O}_{\mathbb{P}^{n+1}}(2)|$ with $\widehat{L} \cong \mathcal{O}_{\mathbb{P}^{n+1}}(1)_{\widehat{X}}$; or*
3. *a scroll over a curve, i.e., there exists a morphism with connected fibers $\varphi : \widehat{X} \to C$ from $\widehat{X}$ onto a smooth curve $C$, and with $K_{\widehat{X}} + n\widehat{L}$ the pullback of an ample and spanned line bundle on $C$.*

The morphism $\phi : \widehat{X} \to \mathbb{P}^{h^0(K_{\widehat{X}}+(n-1)\widehat{L})-1}$ is called the *first adjunction mapping.* The structure of $\phi$ is given by the following theorem. Besides [12], I refer to Sommese [31, 32, 33, 34, 35].

**Theorem 1.3** *Let $\widehat{L}$ be a very ample line bundle on a complex connected $n$-dimensional projective manifold $\widehat{X}$. Assume that $K_{\widehat{X}} + (n-1)\widehat{L}$ is spanned. Let $\phi = s \circ r$ denote the Remmert-Stein factorization of $\phi$ with $r : \widehat{X} \to X$ having connected fibers with $X$ normal and $s : X \to \mathbb{P}^{h^0(K_{\widehat{X}}+(n-1)\widehat{L})-1}$ finite. Then the following are all the possibilities.*

1. *$-K_{\widehat{X}} := (n-1)\widehat{L}$: these are called Del Pezzo manifolds and they are completely classified, see Fujita [25]; or*

2. *if* $\dim X = 1$ *then* $(\widehat{X}, \widehat{L})$ *is called a quadric fibration: in this case all fibers are irreducible quadrics, i.e., given a fiber* $F$ *of* $r$, $F$ *is isomorphic to a quadric in* $\mathbb{P}^n$ *with at most one isolated singularity, and* $L_F$ *is the restriction of* $\mathcal{O}_{\mathbb{P}^n}(1)$ *under this embedding; or*

3. *if* $\dim X = 2$ *then* $(\widehat{X}, \widehat{L})$ *is a scroll over a surface: in this case* $r$ *is a* $\mathbb{P}^{n-2}$*-bundle with* $L_F \cong \mathcal{O}_{\mathbb{P}^{n-2}}(1)$ *for any fiber* $F$ *of* $r$*; or*

4. $r$ *is birational and has a smooth* $3$*-dimensional image* $X$ *with* $r : \widehat{X} \to X$ *the blowing up of* $X$ *on a finite set* $F$*. In this case the bundle* $L := (\phi_* L)^{**}$ *is ample and* $\widehat{L} \cong \phi^* \widehat{L} - \phi^{-1}(F)$ *(or equivalently* $K_{\widehat{X}} + (n-1)\widehat{L} \cong \phi^*(K_X + (n-1)L)$*).*

*When* $r$ *is birational, the pair* $(X, L)$ *is called the* first reduction *of* $(\widehat{X}, \widehat{L})$.

The major fact about the first adjunction mapping in the generic situation, when $r$ is birational, is the following. Besides [12], I refer to Serrano [29] and Sommese and Van de Ven [39].

**Theorem 1.4** *Let* $\widehat{L}$ *be a very ample line bundle on a complex connected* $n$*-dimensional projective manifold* $\widehat{X}$*. Assume that* $K_{\widehat{X}} + (n-1)\widehat{L}$ *is spanned with* $\phi = s \circ r$ *as above. If* $r$ *is birational then* $s$ *is an embedding, i.e.,* $K_X + (n-1)L$ *is very ample.*

The above result shows that the first reduction $(X, L)$ is very simply related to $(\widehat{X}, \widehat{L})$. The first reduction and the first adjunction mapping have proved very useful in the study of the geometry of the pair $(\widehat{X}, \widehat{L})$. They give a strong relation between the geometry of $(\widehat{X}, \widehat{L})$ and the geometry of the curve sections of $(\widehat{X}, \widehat{L})$. This stems from the fact that except for scrolls over curves, $K_{\widehat{X}} + (n-1)\widehat{L}$ is the unique line bundle on $\widehat{X}$, which, for each curve section, restricts to the canonical bundle of the curve section. Since only a codimension $h^1(\mathcal{O}_X)$ subspace of the sections of the canonical bundle of a curve section of $(\widehat{X}, \widehat{L})$ extend to $K_{\widehat{X}} + (n-1)\widehat{L}$, it is remarkable that $K_{\widehat{X}} + (n-1)\widehat{L}$ is so well behaved. In analogy with the first adjunction map we would like to have a map closely associated to the canonical maps of the smooth surfaces in $|\widehat{L}|$. The following problem is the central topic of this survey. It will be gradually refined below (see Problems 1.8 and 1.11).

**Problem 1.5** *Assume that* $\widehat{L}$ *is a very ample line bundle on an connected* $n$*-dimensional projective manifold* $\widehat{X}$*. For each* $k \geq 1$ *decide whether the mapping*

$$\widehat{\Phi}_k : \widehat{X} \to \mathbb{P}^{h^0(k(K_{\widehat{X}} + (n-2)\widehat{L})) - 1}$$

*associated to* $|k(K_{\widehat{X}} + (n-2)\widehat{L})|$ *exists and if it exists work out the structure of* $\widehat{\Phi}_k$*.*

The explicit description above of all the pairs $(\hat{X}, \hat{L})$ when $K_{\hat{X}} + (n-1)\hat{L}$ has no sections shows that, with the exception of scrolls over smooth surfaces, surface sections $\hat{S}$ of $(\hat{X}, \hat{L})$ are of negative Kodaira dimension and thus $h^0(k(K_{\hat{X}} + (n-2)\hat{L})) = 0$ for any $k > 0$. In the case of scrolls over surfaces the bundle $k(K_{\hat{X}} + (n-2)\hat{L})$ is negative restricted to any fiber, and therefore in this case $h^0(k(K_{\hat{X}} + (n-2)\hat{L})) = 0$ for any $k > 0$. Therefore, in the study of the mappings $\hat{\Phi}_k$ associated to $|k(K_{\hat{X}} + (n-2)\hat{L})|$, we can restrict ourselves without loss of generality to pairs $(\hat{X}, \hat{L})$ with a first reduction $(X, L)$. Here a key result is the following, which was proved first by this author [33, 35] for threefolds with $\hat{L}$ merely ample with one smooth $\hat{S} \in |\hat{L}|$ (see also Fania and Sommese [22]), and later generalized under a variety of assumptions by Fujita [24], Ionescu [26], Sommese [37], Beltrametti and Sommese [10], and others. For an exposition of the known results in this direction see [12, Chapter 7].

**Theorem 1.6** *Assume that $\hat{L}$ is a very ample line bundle on a smooth projective $n$-fold $\hat{X}$. Assume that $(X, L)$, the first reduction of $(\hat{X}, \hat{L})$, exists. Then either:*

1. *$(X, L) \cong (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3))$;* *or*
2. *$(X, L) \cong (\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2))$;* *or*
3. *$(X, L) \cong (Q, \mathcal{O}_Q(2))$ where $(Q, \mathcal{O}_Q(1)$ is a quadric in $\mathbb{P}^4$;* *or*
4. *$X$ is a $\mathbb{P}^2$-bundle $f : X \to Y$ over a smooth curve $Y$ with $2K_X + 3L \cong f^*H$ for an ample line bundle $H$ on $Y$;* *or*
5. *$K_X + (n-2)L$ is nef, i.e., for all effective curves $C \subset X$ we have $(K_X + (n-2)L) \cdot C \geq 0$.*

In all the cases of this theorem except the last when $K_X + (n-2)L$ is nef, the surfaces $\hat{S} \in |\hat{L}|$ are of negative Kodaira dimension. Thus, up to a very explicit list of pairs for which $k(K_{\hat{X}} + (n-2)\hat{L})$ has no sections for any $k > 0$, we have that $K_X + (n-2)L$ is nef. Therefore in the study of $\hat{\Phi}_k$, we can without loss of generality assume that $K_X + (n-2)L$ is nef. We often use the notation $\mathcal{K}$ for the bundle $K_X + (n-2)L$. We define the *second adjunction mapping* as the meromorphic map $\Phi_1$ associated to $|\mathcal{K}|$ when $h^0(\mathcal{K}) > 0$. More generally for $k > 0$ we define the map $\Phi_k$ as the meromorphic mapping associated to $|k\mathcal{K}|$. Let $\hat{X}, \hat{L}, X, L, \phi, F$ be as in part 4 of Theorem 1.3. It is a result of this author [38, Lemma (0.3.1)] that the biholomorphism of $X - F$ with $\hat{X} - \phi^{-1}(F)$ gives a canonical isomorphism $\phi_* : H^0(k(K_{\hat{X}} + (n-2)\hat{L})) \to H^0(k\mathcal{K})$ for all $k > 0$. Thus $\hat{\Phi}_k$ exists for a given $k > 0$ if and only if $\Phi_k$ exists for the same $k > 0$. Moreover $\hat{\Phi}_k$ factors $\Phi_k \circ \phi$.

$$\hat{X} \xrightarrow{\phi} X \xrightarrow{\Phi_k} \mathbb{P}^{h^0(k\mathcal{K})-1} \tag{1}$$

Since the cases when either the first reduction does not exist or the first reduction $(X, L)$ exists but $\mathcal{K}$ is not nef are completely classified, we lose nothing by working on the first reduction $(X, L)$ with $\mathcal{K}$ nef. In all cases $\mathcal{K}$ is the only line bundle on $X$ with the property that the restriction to smooth surface sections $S$ of $(X, L)$ is the canonical bundle of $S$. This follows from the adjunction formula $\mathcal{K}_S \cong K_S$, and the first Lefschetz theorem, which implies that restriction gives an injection $\mathrm{Pic}(X) \to \mathrm{Pic}(S)$. Note that the above theorems imply that the smooth surface sections of $(\hat{X}, \hat{L})$ are in one-to-one correspondence with the smooth surface sections of $(X, L)$ that contain $F$. The nefness of $\mathcal{K}$, i.e., $K_X + (n-2)L$, implies that the surface sections of $(\hat{X}, \hat{L})$ are of nonnegative Kodaira dimension and $\phi_{\hat{S}} : \hat{S} \to S$, the first reduction mapping restricted to a smooth surface section $\hat{S}$ of $(\hat{X}, \hat{L})$, is the map from $\hat{S}$ to its minimal model $S$. It is worth emphasizing that it is quite hard for a surface $\hat{S}$ of nonnegative Kodaira dimension to be a surface section of a projective manifold, $(\hat{X}, \hat{L})$. For example, if the map from $\hat{S}$ to $S$, the minimal model of $\hat{S}$, is not a simple blowup of a finite set, then the full result, of which a brief summary is given above, implies that $\hat{X}$ is a $\mathbb{P}^{n-2}$-bundle over a surface with $\hat{S}$ a meromorphic section. The same result holds (see Sommese [36]) if there are more than four irreducible curves in a connected component of the set of smooth rational curves with self-intersection $-2$ on the minimal model of $\hat{S}$.

Since $\mathcal{K}$ is nef, it is a consequence of the Kawamata-Shokurov basepoint free theorem that all large positive multiples of $\mathcal{K}$ are spanned. By considering the Remmert-Stein factorization of the morphism associated to the sections of any positive multiple of $\mathcal{K}$ that is spanned we obtain the following structure theorem.

**Theorem 1.7** *There is a holomorphic map $r : X \to Y$ with connected fibers onto a normal projective variety $Y$, and an ample line bundle $\mathcal{H}$ on $Y$ such that $\mathcal{K} \cong r^*\mathcal{H}$. The triple $(Y, \mathcal{H}, r)$ depends only on $(X, L)$, i.e., $(Y, \mathcal{H}, r)$ is independent of the positive spanned multiple of $\mathcal{K}$ used. The dimension of $Y$ is equal* $0, 1, 2, 3, n$.

We define $d_i = \mathcal{K}^i \cdot L^{n-i}$ for $i = 0, 1, \ldots, n$. Following tradition we usually use $d$ instead of $d_0$. Let Since $\mathcal{K}$ is nef and some power is spanned, we see that

$$d_i \geq 0 \ \text{ with strict inequality if } \ \dim r(X) \geq i. \tag{2}$$

The $d_i$ are crucial invariants in the understanding of lower bounds for numbers of sections of $\mathcal{K}$, and the properties of the maps, e.g., of the degrees of the maps $s_k$ and the spaces $Z_k$ occurring in the factorization (6) below. M. Beltrametti and this author have recently made a systematic investigation of these numbers [16] and as a consequence have been able to show [17] the sharpened degree bounds for the map $s_k$ described later in this paper.

In light of the above we can rephrase Problem 1.5 as follows.

**Problem 1.8** *Assume that $\hat{L}$ is a very ample line bundle on an connected $n$-dimensional projective manifold $\hat{X}$. Assume that either $\kappa(K_{\hat{X}}+(n-2)\hat{L}) \geq 0$ or equivalently that the first reduction $(X, L)$ of $(\hat{X}, \hat{L})$ exists with $\mathcal{K} := K_X + (n-2)L$ nef. For each $k \geq 1$ decide whether the mapping $\Phi_k$ associated to $|k\mathcal{K}|$ exists and if it exists work out the structure of $\Phi_k$.*

In [38, Theorem (2.0)] this author showed that we have a complete and optimal answer to the existence of the mapping $\Phi_k$.

**Theorem 1.9** *Let $\hat{L}$ be a very ample line bundle on a connected $n$-dimensional projective manifold $\hat{X}$. The following are equivalent:*

1. $h^0(K_{\hat{X}} + (n-2)\hat{L}) > 0$;
2. $\kappa(K_{\hat{X}} + (n-2)\hat{L}) \geq 0$, *i.e.,* $h^0(k(K_{\hat{X}} + (n-2)\hat{L})) > 0$ *for some* $k > 0$;
3. *the first reduction* $\phi : (\hat{X}, \hat{L}) \to (X, L)$ *exists with* $\mathcal{K}$ *nef.*

As pointed out earlier $H^0(k(K_{\hat{X}} + (n-2)\hat{L}))$ and $H^0(k\mathcal{K})$ are canonically isomorphic for $k \geq 1$. This result is in fact true for ample and spanned line bundles on Gorenstein threefolds with isolated singularities or for smooth ample divisors on smooth projective threefolds. Theorem 1.9 follows from equation 2 combined with the following result in this author's paper [38]. Equation 3 below follows by means of a covering trick from an inequality of Miyaoka, and equation 4 below follows from a log Chern inequality of Tsuji.

**Theorem 1.10** *Let $L$ be an ample line bundle on a Gorenstein projective threefold. Assume that $\mathcal{K}$ is nef. If $S \in |L|$ is smooth and $L$ is spanned, then*

$$h^0(\mathcal{K}) \geq \frac{\chi(\mathcal{O}_S)}{2} + \frac{d_1}{36}. \tag{3}$$

*If $L$ is not necessarily spanned, but $X$ and $S \in |L|$ are smooth,*

$$h^0(\mathcal{K}) \geq \frac{\chi(\mathcal{O}_S)}{2} + \frac{d_1}{24} + \frac{d_3}{64}. \tag{4}$$

*From here on we make the assumption that the first reduction $(X, L)$ exists and $\mathcal{K} := K_X + (n-2)L$ is nef.*

By Theorem 1.7, the mapping $\Phi_k$ factors as $\Phi_k = \psi_k \circ r$ giving the factorization $\hat{\Phi}_k = \psi_k \circ r \circ \phi$

$$\hat{X} \xrightarrow{\phi} X \xrightarrow{r} Y \xrightarrow{\psi_k} \mathbb{P}^{h^0(k\mathcal{K})-1} \tag{5}$$

where $\psi_k : Y \to \mathbb{P}^{h^0(k\mathcal{H})-1}$ is the meromorphic mapping associated to $|k\mathcal{H}|$. We can Remmert-Stein factorize $\psi_k = s_k \circ \mathcal{R}_k$ where $\mathcal{R}_k : Y \to Z_k$ is a meromorphic map having connected fibers with $\mathcal{R}_k$'s generic fiber irreducible and where $s_k : Z_k \to s_k(Z_k) \subset \mathbb{P}^{h^0(k\mathcal{K})-1}$ is a finite mapping. This gives the factorization $\hat{\Phi}_k = s_k \circ \mathcal{R}_k \circ r \circ \phi$

$$\hat{X} \xrightarrow{\phi} X \xrightarrow{r} Y \xrightarrow{\mathcal{R}_k} Z_k \xrightarrow{s_k} \mathbb{P}^{h^0(k\mathcal{K})-1} \tag{6}$$

into maps that our results describes. Therefore we can refine the Problem 1.5 as follows.

**Problem 1.11** *Assume that $\hat{L}$ is a very ample line bundle on an connected $n$-dimensional projective manifold $\hat{X}$. Assume that either $\kappa(K_{\hat{X}} + (n-2)\hat{L}) \geq 0$ or equivalently that the first reduction $(X, L)$ of $(\hat{X}, \hat{L})$ exists with $\mathcal{K} := K_X + (n-2)L$ nef.*

1. *Classify all pairs $(\hat{X}, \hat{L})$ for which $k\mathcal{K}$ is not spanned.*
2. *For each $k \geq 1$ work out the properties of the map $\mathcal{R}_k$, e.g.,*
   (a) *what is the dimension of $Z_k$; and*
   (b) *how far is $\psi_k$ from being a morphism?*
3. *For each $k \geq 1$ work out the properties of the map $s_k$, e.g.,*
   (a) *what is the degree of $s_k$; and*
   (b) *how far is $s_k$ from being birational?*

We will see below that:

1. We have that $k\mathcal{K}$ is spanned for $k \geq 2$ (the hard case when $\kappa(\mathcal{K}) \geq 3$ was shown by this author [38]).
2. $\psi_k$ is an isomorphism for $k \geq 3$.
3. In the case of $k = 2$ we have strong results on the degree of $s_2$, e.g., under weak conditions it is birational.
4. We have significant results about $\mathcal{R}_1$ under the assumption that
$$\dim Z_1 \leq 3.$$
5. There are exceptions (Theorem 4.1 below) to the spannedness of $\mathcal{K}$.

The following lemma shows why it is often straightforward to lift results about $K_X + (n-2)L$ from threefold sections of $(\hat{X}, \hat{L})$ to $\hat{X}$.

**Lemma 1.12** *Let $\mathcal{L}$ be an ample line bundle on an $n$-dimensional projective manifold $\mathcal{X}$. Assume that $n \geq 4$ and that $A_1, \ldots, A_{n-3}$ are smooth elements of $|\mathcal{L}|$ that meet transversely in a smooth threefold $A$. Then the map $H^0(K_{\mathcal{X}} + (n-2)\mathcal{L}) \to H^0(K_A + \mathcal{L}_A)$ is onto.*

**Proof.** For simplicity assume that $n = 4$. Consider the residue sequence:

$$0 \longrightarrow K_{\mathcal{X}} + \mathcal{L} \longrightarrow K_{\mathcal{X}} + 2\mathcal{L} \longrightarrow K_A + \mathcal{L}_A \longrightarrow 0.$$

The lemma follows from Kodaira's vanishing theorem. Q.E.D.

**Remark 1.13** The above result is often used in conjunction with the fact that through any two points or through a tangent direction at a point we can always find a smooth threefold section of $(\widehat{X}, \widehat{L})$ where $\widehat{L}$ is a very ample line bundle on a smooth projective manifold of dimension $n \geq 4$.

**Remark 1.14** Let $(\widehat{X}, \widehat{L})$ as in Problem 1.11. Let $A$ be a general threefold section of $(\widehat{X}, \widehat{L})$. If $\widehat{\Phi}_1$ has an image of dimension $\leq 3$ then $\widehat{\Phi}_1(\widehat{X}) = \widehat{\Phi}(A)$. Moreover if $\widehat{\Phi}_1$ has an image of dimension $= 3$ then the degree of the map associated to $|K_A + \widehat{L}_A|$ is a positive multiple of the degree of $s_1$. If $\widehat{\Phi}_1$ has an image of dimension $\leq 2$ and is not holomorphic then the map associated to $|K_A + \widehat{L}_A|$ is not holomorphic. Also in this case since a general fiber of $\mathcal{R}_1$ is at least two dimensional and since $A$ meets a general fiber of $\mathcal{R}_1$ in an irreducible set, it follows that the degree of the finite map arising from the Remmert-Stein factorization of the map associated to $|K_A + \widehat{L}_A|$ equals the degree of $s_1$.

## 2 Spannedness results

In this section we discuss what we know about the spannedness of $\mathcal{K}$.

I know few general conditions guaranteeing the spannedness of $\mathcal{K}$. Here is one result in this direction based on the generalization of Reider's theorem due to Ein and Lazarsfeld [20].

**Theorem 2.1 (Beltrametti, Schneider, and Sommese [7])** *Let $\mathcal{L}$ be a very ample line bundle on a connected $n$-dimensional projective manifold $\mathcal{X}$ with $n \geq 3$. Assume that the first reduction $(\mathcal{X}', \mathcal{L}')$ exists, that $\mathcal{L}'^n \geq 850$, and that there are no nontrivial maps of $\mathbb{P}^1$ to threefold sections $A$ of $(\mathcal{X}', \mathcal{L}')$, e.g., either $T_A^*$ is nef or that $A$ is hyperbolic. Then it follows that $K'_{\mathcal{X}} + (n-2)\mathcal{L}'$ is spanned.*

I refer also to [7, 8] for a number of results on the spannedness of $K_{\mathcal{X}} \otimes \det \mathcal{V}$ where $\mathcal{V}$ is a vector bundle on $\mathcal{X}$ satisfying appropriate ampleness conditions. It is a result of Beltrametti and this author (combine Lemma 2.2 and Proposition 3.1 of [11]) that given a line bundle $\mathcal{L}$ on a connected $n$-dimensional projective manifold $\mathcal{X}$ that is 3-jet ample in the sense of [11], then $K_{\mathcal{X}} + (n-2)\mathcal{L}$ is spanned for $n \geq 3$ unless $(\mathcal{X}, \mathcal{L}) = (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3))$. A generalization of this result follows easily from some results of Lanteri, Palleschi, and this author [27]. For a point $x$ of a variety $V$, let $\mathfrak{m}_x$ denote the subsheaf of $\mathcal{O}_V$ which is equal to $\mathcal{O}_V$ away from $x$ and is equal to the maximal ideal of $\mathcal{O}_V$ at $x \in V$.

**Theorem 2.2 (Lanteri, Palleschi, and Sommese)** *Assume that $\mathcal{L}$ is a line bundle on a connected $n$-dimensional projective manifold $\mathcal{X}$. Assume that for some $k > 0$ and each $x \in \mathcal{X}$, $x$ is an isolated point in the basepoint locus of $|\mathcal{L} \otimes \mathfrak{m}_x^k|$. If $t$ is an integer satisfying $\mathcal{L}^n \geq \frac{n^n + 1}{t^n}$ and $t \geq \frac{n}{k}$ then $K_{\mathcal{X}} + t\mathcal{L}$ is spanned.*

Note that the condition that for each $x \in \mathcal{X}$, $x$ is an isolated point in the basepoint locus of $|\mathcal{L} \otimes \mathfrak{m}_x^k|$, implies the nefness of $\mathcal{L}$. Note also that the above conditions apply to the case of $\mathcal{L} = g^*H$ where $g : \mathcal{X} \to \mathcal{Y}$ is a finite-to-one morphism from a projective manifold onto a variety $\mathcal{Y}$, and $H$ is $(k-1)$-jet ample unless $g$ is an isomorphism to $\mathbb{P}^n$ with $t\mathcal{L} \cong \mathcal{O}_{\mathbb{P}^n}(n)$. The most interesting cases of this result for this article occur when $(n, k, t) = (3, 3, 1)$ and $(n, k, t) = (4, 2, 2)$. For example, here is the three-dimensional case.

**Corollary 2.3 (Lanteri, Palleschi, and Sommese)** *Assume that $\mathcal{L}$ is a line bundle on a connected three-dimensional projective manifold $\mathcal{X}$ with $\mathcal{L}^n \geq 28$. If, for some $k \geq 3$ and each $x \in \mathcal{X}$, $x$ is an isolated point in the basepoint locus of $|\mathcal{L} \otimes \mathfrak{m}_x^k|$, then $K_{\mathcal{X}} + \mathcal{L}$ is spanned.*

Here is another variant of the above. Recall [5] that $\mathcal{L}$ is said to be $k$-spanned if given any zero dimensional curvilinear scheme $Z$ on $X$ of length $\leq k + 1$ it follows that the map $H^0(\mathcal{L}) \to H^0(\mathcal{L} \otimes \mathcal{O}_Z)$ is onto.

**Theorem 2.4** *Let $\mathcal{L}$ be a 3-spanned line bundle on a connected $n$-dimensional projective manifold $\mathcal{X}$ with $n \geq 3$. Assume that $\mathcal{L}^n \geq 850$. Then $K_{\mathcal{X}} + (n-2)\mathcal{L}$ is spanned by global sections.*

**Proof.** Using Lemma 1.12 and Remark 1.13 we see that we are reduced to the case of $n = 3$. By [20, Theorem 1*], it suffices to show that given any irreducible curve $C \subset \mathcal{X}$ and any irreducible divisor $D \subset \mathcal{X}$, we have $\mathcal{L} \cdot C \geq 3$ and $\mathcal{L}^2 \cdot D \geq 5$. Note that by the 3-spannedness of $\mathcal{L}_C$ it follows that we can separate any four points of $C$, which implies that $\mathcal{L} \cdot C \geq 3$. Let $C \subset D$ denote a general element of $|\mathcal{L}_D|$. We know that $C$ is irreducible since $D$ is irreducible and $\mathcal{L}$ is very ample. Since we can separate up to at least four points of $C$, we see that $|\mathcal{L}_C|$ embeds $C$ in $\mathbb{P}^N$ with $N \geq 3$. By Castelnuovo's bound on the genus of an irreducible curve we see that $\delta := \mathcal{L}^2 \cdot D \leq 4$ implies that the arithmetic genus of $C$ is at most 1. Choosing a set $\mathcal{F}$ of $\delta - 1 \leq 3$ distinct smooth points of $C$, we have the result that $\mathcal{L}_C - \mathcal{F}$ is spanned. If the genus of $C$ is 1, then we have the absurdity that the degree of $\mathcal{L}_C - \mathcal{F}$ is 1. Thus we can assume without loss of generality that $C$ is genus 0 and hence smooth. This implies that $C$ did not meet the singularities of $D$ and thus $D$ has at most isolated singularities. Since $D$ is a divisor on a smooth threefold, it follows that $D$ is normal. Using the classification of normal surfaces with a rational curve as ample divisor, we see that either

1. $(D, \mathcal{L}_D) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(k))$ with $k = 1, 2$; or
2. $D$ is a $\mathbb{P}^1$-bundle $p : D \to \mathbb{P}^1$ over $\mathbb{P}^1$ with $\mathcal{L}_f \cong \mathcal{O}_{\mathbb{P}^1}(1)$ for a fiber $f$ of $p$; or
3. $(D, \mathcal{L}_D)$ is a cone over $\mathbb{P}^1$, i.e., $D$ is $\hat{D} := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(k))$ for some $k \geq 1$ with the exceptional section corresponding to the surjection $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(k) \to \mathcal{O}_{\mathbb{P}^1}$ blown down and with $\mathcal{L}_D$ the unique line bundle that pulls back to the tautological line bundle on $\hat{D}$.

In any of these cases $D$ contains curves $E$ with $\mathcal{L} \cdot E \leq 2$, contradicting the fact that $\mathcal{L} \cdot C \geq 3$ for all curves on $X$. Q.E.D.

## 3 The second adjunction mapping: the case when $\kappa(\mathcal{K}) \leq 2$

We assume throughout this section that $\hat{L}$ is a very ample line bundle on an $n$-dimensional projective manifold $\hat{X}$ with $n \geq 3$. We assume further that the second reduction $\phi : (\hat{X}, \hat{L}) \to (X, L)$ exists with $\mathcal{K} := K_X + (n-2)L$ nef. We use the notation occurring in factorization (6).

There are four cases, $\kappa(K_{\hat{X}} + (n-2)\hat{L}) = \kappa(\mathcal{K}) = \dim Y = 0, 1, 2$. Since under these assumptions any threefold section of $(X, L)$ surjects onto the $Y$, we can by using Lemma 1.12 reduce to the case of $n = 3$. In the case when $\dim Y = 0$ we see from Theorem 1.7 that $k\mathcal{K}$ is trivial for all $k > 0$, and in particular $k\mathcal{K}$ is spanned for all $k > 0$.

### 3.1 The case of $\kappa(\mathcal{K}) = 1$

The first interesting case is when $\dim Y = 1$. In this case $Y$ is a smooth curve. Since $\mathcal{H}$ is ample, we have spannedness of $\mathcal{K}$ (and very ampleness of $\mathcal{H}$) if $Y \cong \mathbb{P}^1$. Further we see that for the general fiber $F$ of $r$ we have $K_F + L_F \cong \mathcal{O}_F$. Thus $(F, L_F)$ is a so-called Del Pezzo surface. These rational surfaces are completely classified, e.g., we have $L_F^2 \leq 9$. From the relative Kodaira vanishing theorem we conclude that $r_{(1)}\mathcal{O}_X$, the first derived functor of the direct image of $\mathcal{O}_X$ with respect to $r$, is 0. From this we conclude that the genus of $Y$ equals the irregularity $q := h^1(\mathcal{O}_X)$ of $X$, which equals $h^1(\mathcal{O}_S)$ for a surface section $S$ of $(X.L)$ by the first Lefschetz theorem. Moreover $r_S : S \to Y$ is an elliptic surface of Kodaira dimension 1. The key to analyzing the structure of $(X, L)$ in this case is to use the canonical bundle formula on $S$ (see [1, Theorem (12.1)]). This was done in [33, 38] and the conclusions are that $r_S$ has no multiple fibers and, by using the injection of $\mathrm{Pic}(X) \to \mathrm{Pic}(S)$, that $\deg \mathcal{H} = 2q - 2 + \chi(\mathcal{O}_S)$. It is well known that a line bundle $\mathcal{E}$ on a Riemann surface of positive genus $q$ is spanned if $\deg \mathcal{E} \geq 2q$, and is very ample if $\deg \mathcal{E} \geq 2q + 1$. Thus we conclude that $\mathcal{K}$ is spanned if $\chi(\mathcal{O}_S) \geq 2$ and $\mathcal{H}$ is very ample if $\chi(\mathcal{O}_S) \geq 3$. We need a Theorem, which seems to be new in the case $\chi(\mathcal{O}_S) = 0$. The argument we give works without change for merely ample line bundles with at least one smooth divisor in the linear system of the line bundle. We will need a very useful result from my first paper on hyperplane sections [30, 33].

**Theorem 3.1** *Let $H$ be an ample line bundle on a connected projective manifold, $V$. Let $A$ be a $k$-fold section of $(V, H)$. If $k \geq 2$, then the universal cover of $A$ is not contractible, i.e., $A$ is not a $K(\pi, 1)$. In particular if $A$ is a smooth general type surface, then $K_A^2 \leq 9\chi(\mathcal{O}_A) - 1$.*

The last statement of the result follows since equality in Miyaoka's inequality for general type surfaces implies that the universal cover of the surface is biholomorphic to the unit ball in $\mathbb{C}^2$.

**Theorem 3.2** *If $\hat{L}$ is a very ample line bundle on a smooth projective threefold $\hat{X}$ and if $\chi(\mathcal{O}_{\hat{S}}) \leq 0$ for a smooth $\hat{S} \in |\hat{L}|$, then $\kappa(K_{\hat{X}} + \hat{L}) = -\infty$.*

**Proof.** If $\kappa(K_{\hat{X}} + \hat{L}) \geq 0$ then we are in the situation of Theorem 1.7. Thus $\mathcal{K}$ is nef and hence $K_S$ is nef for smooth $S \in |L|$. Since $\kappa(\hat{S}) = -\infty$ if $\chi(\mathcal{O}_{\hat{S}}) < 0$, we can assume without loss of generality that $\chi(\mathcal{O}_{\hat{S}}) = 0$. In case $\dim r(X) = 0$ it follows from the residue sequence

$$0 \to K_X \to \mathcal{K} \to K_S \to 0$$

that $\chi(\mathcal{O}_S) = \chi(\mathcal{K}) + \chi(\mathcal{O}_X)$. Since $K_X + L \cong \mathcal{O}_X$ we conclude from the Kodaira vanishing theorem that $\chi(\mathcal{O}_S) = 2$ (in fact, $S$ is a $K3$ surface). So $\chi(\mathcal{O}_{\hat{S}}) \neq 0$ in this case. If $\dim r(X) \geq 2$ then $K_S^2 = \mathcal{K}^2 \cdot L \geq 1$ and thus we conclude that $S$ is of general type. This implies that $\chi(\mathcal{O}_S) > 0$. Thus we have $\dim r(X) = 1$. Since $\chi(\mathcal{O}_S) = 0$, we conclude from $2q - 2 + \chi(\mathcal{O}_S) = \deg \mathcal{H} > 0$ that $q \geq 2$. Since $\chi(\mathcal{O}_S) = 0$ and since $K_S^2 = \mathcal{K}^2 \cdot L = 0$, it follows from Noether's formula that the Euler characteristic of $S$ is 0. Since $r : S \to Y$ has no multiple fibers, we conclude from Corollary (11.6) and Remark (11.5) of [1] that $r$ has no singular fibers and thus that $r$ is a $C^\infty$ fiber bundle over a positive genus curve. Thus the universal cover of $S$ is contractible. This is not possible by Theorem 3.1. Q.E.D.

Note that this result contains a nonexistence result as corollary.

**Corollary 3.3** *Let $\mathcal{L}$ be a very ample line bundle on a smooth $n$-dimensional projective manifold $\mathcal{X}$ with $n \geq 3$. If $h^0(K_S) = 0$ for a smooth surface section $S$ of $(\mathcal{X}, \mathcal{L})$, and $S$ has nonnegative Kodaira dimension, then it follows that $\mathcal{X}$ is a $\mathbb{P}^{n-2}$-bundle over a smooth surface $\mathcal{Y}$ with $S$ a meromorphic section over $\mathcal{Y}$.*

The very ampleness can be relaxed to the assumption that $\mathcal{L}$ is ample and there are $n - 2$ elements of $|\mathcal{L}|$ meeting transversely in a smooth surface $S$. Using Theorem 3.2 we see that $\mathcal{H}$ and (hence) $\mathcal{K}$ are spanned except possibly if $\chi(\mathcal{O}_S) = 1$, and $\mathcal{H}$ is very ample except possibly if $\chi(\mathcal{O}_S) = 1, 2$.

For $k \geq 2$, $k\mathcal{K}$ is spanned. For $k \geq 2$, $k\mathcal{H}$ is very ample, with the possible exception if $k = 2$ and $h^1(\mathcal{O}_S) = \chi(\mathcal{O}_S) = 1$. If $k = 2$ and $h^1(\mathcal{O}_S) = \chi(\mathcal{O}_S) = 1$, then the degree of the mapping $s_2$ is two. Thus the following is all that remains unanswered when $\kappa(K_{\hat{X}} + (n-2)\hat{L}) = 1$.

**Problem 3.4** *Let $\hat{L}$ be a very ample line bundle on an $n$-dimensional connected projective manifold $\hat{X}$ with $n \geq 3$. Assume that $\kappa(K_{\hat{X}} + (n-2)\hat{L}) = 1$. Let $(X, L)$ be the first reduction of $(\hat{X}, \hat{L})$ with $\mathcal{K}$ nef, and $\mathcal{H}$, $Y$, and $q = h^1(\mathcal{O}_{\hat{X}})$ as above.*

1. *Enumerate the exceptions (if there are any) to $\mathcal{K}$ being spanned when $\chi(\mathcal{O}_S) = 1 \leq h^1(\mathcal{O}_S)$.*
2. *Enumerate the exceptions (if there are any) to $\mathcal{H}$ being very ample when $\chi(\mathcal{O}_S) = 1, 2;\ \ h^1(\mathcal{O}_S) \geq 1$ and to $2\mathcal{H}$ being very ample when $h^1(\mathcal{O}_S) = \chi(\mathcal{O}_S) = 1$.*

## 3.2 The case of $\kappa(\mathcal{K}) = 2$

In this case the general fiber of $r : X \to Y$ is a smooth quadric. Beltrametti and this author [9] showed that when $n \geq 4$ the fibers of the map $r$ are all of pure dimension $n-2$ and when $n = 3$ they classified the two-dimensional fibers of such morphisms. Besana [19] showed that $Y$ was a smooth surface and that $\mathcal{H} = K_Y + \mathcal{E}$ for an ample line bundle $\mathcal{E}$ on $Y$. Using this and Reider's theorem Besana showed spannedness of $\mathcal{K}$ except if $\mathcal{E}^2 \leq 4$ (in fact Besana shows the more precise result that $\mathcal{K}$ is spanned except if $\mathcal{E}^2 \leq 4 - u$, where $u$ is the number of divisorial fibers of the map $r$). Since the branch locus of a map $r : S \to Y$ for smooth $S \in |L|$ is in $|2\mathcal{E}|$, this is seen to be a very restrictive condition. In [3], Beltrametti, Besana, and this author showed that $h^0(\mathcal{K}) \geq 2$ with a few possible exceptions. It was noted [15, Remark 1.5] that an argument using Reider's theorem shows that $2\mathcal{K}$ is always spanned. Here is a similar argument showing that $3\mathcal{H}$ is very ample on $Y$.

**Theorem 3.5** *$3\mathcal{H}$ is very ample on $Y$.*

**Proof.** By the result of Besana mentioned above, we know that $\mathcal{H} = K_Y + \mathcal{E}$ where $Y$ is smooth and $\mathcal{E}$ is ample. Write $3\mathcal{H} = K_Y + \mathcal{L}$ with $\mathcal{L} := 2\mathcal{H} + \mathcal{E}$. Note that $\mathcal{L}^2 \geq 13$. Indeed since $\mathcal{H}$ and $\mathcal{E}$ are ample we have $\mathcal{H} \cdot \mathcal{E} \geq 1$. Further we have that $\mathcal{H} \cdot \mathcal{E} = (K_Y + \mathcal{E}) \cdot \mathcal{E}$ is even. Thus $\mathcal{L}^2 = 4\mathcal{H}^2 + 4\mathcal{H} \cdot \mathcal{E} + \mathcal{E}^2 \geq 13$. By Reider's theorem we know that if $K_Y + \mathcal{L}$ is not very ample there is an effective curve $C$ on $Y$ satisfying $\mathcal{L} \cdot C - 2 \leq C^2 \leq \mathcal{L} \cdot C/2 < 2$. Noting that $3 \geq \mathcal{L} \cdot C = (2\mathcal{H} + \mathcal{E}) \cdot C \geq 3$, we see that $\mathcal{L} \cdot C = 3$ and $C^2 = 1$, which by the Hodge index theorem gives the contradiction that $\mathcal{L}^2 \leq 9$. Q.E.D.

The following lemma summarizes what we know about the degree of the mapping associated to $|2\mathcal{H}|$.

**Lemma 3.6** *The degree of the morphism associated to $|2\mathcal{H}|$ is at most four, with equality implying that $\chi(\mathcal{O}_S) = h^0(\mathcal{H}) = 1$ and that $d_2 := K_S^2 = 2, 4$.*

**Proof.** Note that

$$h^0(2\mathcal{H}) = h^0(K_Y + \mathcal{E} + \mathcal{H}) = \chi(2\mathcal{H}) = \mathcal{H}^2 + \frac{(K_Y + \mathcal{E}) \cdot \mathcal{E}}{2} + h^0(\mathcal{H}). \quad (7)$$

Since both $\mathcal{E}$ and $\mathcal{H}$ are ample we know that $\frac{(K_Y + \mathcal{E}) \cdot \mathcal{E}}{2} \geq 1$. Thus equation 7 combined with $h^0(\mathcal{H}) \geq 1$ lets us conclude that $h^0(2\mathcal{H}) \geq \mathcal{H}^2 + 2$. Let $t$ equal the degree of the morphism associated to $|2\mathcal{H}|$. We know that $(2\mathcal{H})^2 = 4\mathcal{H}^2 = t\delta$ where $\delta$ denotes the degree of the image of $Y$ under $|2\mathcal{H}|$. Since $\delta \geq h^0(2\mathcal{H}) - 2$, we conclude that $t \leq 4$ with equality implying that $\mathcal{H} \cdot \mathcal{E} = (K_Y + \mathcal{E}) \cdot \mathcal{E} = 2$ and $h^0(\mathcal{H}) = 1$. Using [3, Proposition (1.1) and Theorem (2.1)] we conclude that $\chi(\mathcal{O}_S) = 1$ and that $d_2 := K_S^2 = 2, 4$. Q.E.D.

**Theorem 3.7** *If $d_2 \geq 10$ and if $s_2$ is not birational then there is a morphism $g : Y \to R$ of $Y$ to a smooth curve $R$ with all fibers irreducible curves of arithmetic genus 1. Moreover the morphism $g$ has a section.*

**Proof.** Note that we have $2\mathcal{H} = K_Y + \mathcal{E} + \mathcal{H}$ and

$$(\mathcal{E} + \mathcal{H})^2 = \mathcal{E}^2 + 2\mathcal{E} \cdot \mathcal{H} + \mathcal{H}^2 = \mathcal{E}^2 + 2\mathcal{E} \cdot (K_Y + \mathcal{E}) + \frac{d_2}{2} \geq 1 + 4 + 5.$$

Therefore given a general point $x \in Y$ and any other different point $y \in Y$, we have that either the map associated to $|2\mathcal{H}|$ separates $x, y$ or there is a curve $C \subset Y$ that contains $x, y$ and such that

$$(\mathcal{E} + \mathcal{H}) \cdot C - 2 \leq C \cdot C < \frac{(\mathcal{E} + \mathcal{H}) \cdot C}{2} < 2.$$

Thus we conclude that $(\mathcal{E} + \mathcal{H}) \cdot C = 2, 3$. In case $(\mathcal{E} + \mathcal{H}) \cdot C = 3$, we get that $C \cdot C = 1$ which gives the contradiction

$$9 = ((\mathcal{E} + \mathcal{H}) \cdot C)^2 \geq \left((\mathcal{E} + \mathcal{H})^2\right)\left(C^2\right) \geq 10.$$

Therefore we conclude that $\mathcal{E} \cdot C = 1$, $\mathcal{H} \cdot C = 1$, $C^2 = 0$. From $\mathcal{E} \cdot C = 1$, we conclude that $C$ is irreducible. From

$$(K_Y + C) \cdot C = (\mathcal{H} - \mathcal{E} + C) \cdot C = 0$$

we conclude that the arithmetic genus of $C$ is 1. Finally note that since $\mathcal{E} \cdot C = 1$ there are a finite number of projective varieties parameterizing curves $C$ satisfying $\mathcal{E} \cdot C = 1$, $\mathcal{H} \cdot C = 1$, $C^2 = 0$. Since there is at least one of these families with a curve passing through a general point of $Y$ and since $C^2 = 0$ we conclude by standard arguments that we have a morphism $g : Y \to R$ to a smooth curve $R$ with the general fiber one of these curves. Since $\mathcal{H}$ is ample and since $\mathcal{H} \cdot f = 1$ for one and hence every fiber of $g$ we conclude that all fibers of $g$ are irreducible. Finally note that since $h^0(\mathcal{H}) \geq 1$ and $\mathcal{H} \cdot f = 1$, there is an irreducible component of a curve in $|\mathcal{H}|$ which is a section of $g$. Q.E.D.

**Corollary 3.8** *Let $\hat{L}$ be a very ample line bundle on an $n$-dimensional projective manifold $\hat{X}$. Assume that $\kappa(K_{\hat{X}} + (n-2)\hat{L}) = 2$. If $d_2 \geq 10$ and $\psi_2$ is not birational, then given a smooth surface section $S$ of $(X, L)$, there is a fibration of $S$ onto a curve $R$ with general fiber a genus two curve which is a double cover of an elliptic curve.*

**Proof.** Lemma 1.12 lets us reduce to the case of $n = 3$. Let $h : S \to R$ be the composition of $r$ with the morphism $g$ of Theorem 3.7. Let $\mathcal{E}$ be as in Theorem 3.7. Note that $S$ is the desingularization of a branched cover of $Y$ with branch locus $B \in |2\mathcal{E}|$. Thus since $\mathcal{E} \cdot f = 1$ for a fiber of $g$ we conclude that a general fiber of $h$ is a double cover of an elliptic curve with branch locus of degree two, and thus a genus two curve. Q.E.D.

We know less about the degree of $\psi_1$, the mapping associated to $\mathcal{H}$.

**Theorem 3.9** *Let $\hat{L}$ be a very ample line bundle on an $n$-dimensional projective manifold $\hat{X}$. Assume that $\kappa(K_{\hat{X}} + (n-2)\hat{L}) = 2$. If $\dim \hat{\Phi}_1(X) = 2$ then the degree of $\psi_1$ is at most 13.*

**Proof.** Let $h := h^0(\mathcal{H})$. We know that $h = h^0(\mathcal{K})$, and since the image of $Y$ under the meromorphic map associated to $\mathcal{H}$ is two-dimensional, we know that $h \geq 3$. Moreover we know that $d_2 := \mathcal{K} \cdot \mathcal{K} \cdot L = 2\mathcal{H}^2$. Thus if the degree of the mapping associated to $|\mathcal{H}|$ is greater than 13, then we have

$$d_2 = 2\mathcal{H}^2 \geq 28(h-2). \tag{8}$$

Using Theorem 3.1 we know that $d_2 \leq 9\chi(\mathcal{O}_S) - 1$ Thus we conclude that $\chi(\mathcal{O}_S) \geq 3h - 5$. Thus by Theorem 1.10 we conclude that

$$h \geq \frac{\chi(\mathcal{O}_S)}{2} + \frac{d_1}{12} \geq \frac{3h-5}{2} + \frac{d_1}{24}.$$

This gives that $5 \geq h + \dfrac{d_1}{6}$. Since $d_1 > 0$ we conclude that $h \leq 4$. We claim that $h \neq 3$. To see this note that if $h = 3$ then by equation 8 we have that $d_2 \geq 28$. Therefore by Miyaoka's inequality for general type surface we conclude that $\chi(\mathcal{O}_S) \geq 4$ for a smooth $S \in |L|$. Thus by Theorem 1.10 we conclude that $d_1 \leq 12$. Therefore by the Hodge inequality we conclude that $d \le \frac{d_1^2}{d_2} \le \frac{144}{28}$, i.e., that $d \le 5$. But this implies by Castelnuovo's inequality implies that $g(C) \le 2$ for a curve section $C$ of $(\hat{X}, \hat{L})$. Since general type surfaces do not contain pencils of curves of genus $\le 1$ we conclude that $g(C) = 2$ and $d = 5$. From this we conclude the absurdity that $d_1 = 2g(C) - 2 - d < 0$. If $h = 4$ then $d_2 \ge 56$ and by Miyaoka's inequality for general type surface we conclude that $\chi(\mathcal{O}_S) \ge 7$. Thus by Theorem 1.10 we conclude that $d_1 \le 12$. Therefore by the Hodge inequality we conclude that $d \le \frac{d_1^2}{d_2} \le \frac{144}{56}$, i.e., that $d \le 2$. This easily contradicts $S$ being a general type surface. Q.E.D.

**Theorem 3.10** *Let $\hat{L}$ be a very ample line bundle on an $n$-dimensional projective manifold $\hat{X}$. Assume that $\kappa(K_{\hat{X}} + (n-2)\hat{L}) = 2$. If $\dim \hat{\Phi}_1(X) = 2$. If $d := L^n \ge 72$, then the degree of $\psi_1$ is at most eight.*

**Proof.** Let $h := h^0(\mathcal{H})$. We know that $h = h^0(\mathcal{K})$, and since the image of $Y$ under the meromorphic map associated to $\mathcal{H}$ is two-dimensional, we know that $h \ge 3$. Moreover we know that $d_2 := \mathcal{K} \cdot \mathcal{K} \cdot L = 2\mathcal{H}^2$. Thus if the degree of the mapping associated to $|\mathcal{H}|$ is greater than 9, then we have

$$d_2 = 2\mathcal{H}^2 \ge 18(h-2).$$

Since $d_2 \le 9\chi(\mathcal{O}_S) - 1$ we conclude that $\chi(\mathcal{O}_S) \ge 2h - 3$. Thus by Theorem 1.10 we conclude that

$$h \ge \frac{\chi(\mathcal{O}_S)}{2} + \frac{d_1}{12} \ge \frac{2h-3}{2} + \frac{d_1}{24}.$$

This gives that $d_1 \le 36$. Since $d_2 \ge 18$, we conclude from the Hodge index theorem that $d \le 72$ with equality implying that $2K_S \sim L$. Thus either $d \ge 73$ and we are done or $d = 72$ and we conclude that $2K_S \sim L$. Using the consequence of the first Lefschetz theorem that the restriction map $\mathrm{Pic}(X) \to \mathrm{Pic}(S)$ is injective for any smooth $S \in |L|$, we conclude that $2\mathcal{K} \sim L$ and therefore that $\mathcal{K}$ is ample. Thus $(X, L)$ can't be a conic fibration over a surface in this case. Q.E.D.

**Example 3.11** Let $(Y, H)$ be the unique example of a smooth surface polarized by a very ample line bundle $H$ such that the morphism associated to $|K_Y + H|$ has degree three. Let $\hat{X} := Y \times \mathbb{P}^1$ and let $a : \hat{X} \to Y$ and $b : \hat{X} \to \mathbb{P}^1$ denote the product projections. Let $\hat{L}$ denote the very ample line bundle, $a^*H \otimes b^*\mathcal{O}_{\mathbb{P}^1}(2)$. Then $K_{\hat{X}} + \hat{L}$ is spanned and the second adjunction mapping of $(\hat{X}, \hat{L})$, i.e., in this case the map associated to $|K_{\hat{X}} + \hat{L}|$, has the Remmert-Stein factorization $a : \hat{X} \to Y$ composed with the three-to-one branched cover associated to $|K_Y + H|$. I know no examples with $s_1$ having higher degree that three.

The following is what remains unanswered when $\kappa(K_{\hat{X}} + (n-2)\hat{L}) = 2$.

**Problem 3.12 (cf. [3])** *Let $\hat{L}$ be a very ample line bundle on an $n$-dimensional projective manifold $\hat{X}$. Assume that $\kappa(K_{\hat{X}} + (n-2)\hat{L}) = 2$. Let $(X, L)$ be the first reduction of $(\hat{X}, \hat{L})$ with $\mathcal{K}$ nef, and with $\mathcal{H}$, $Y$ as above.*

1. *Enumerate the exceptions (if there are any) to $\mathcal{H}$ being spanned when $(\mathcal{H} - K_Y)^2 \leq 4$.*
2. *Enumerate the exceptions (if there are any) to $\mathcal{H}$ and $2\mathcal{H}$ being very ample.*
3. *Find the optimal degree bounds for the maps $s_1, s_2$.*

## 4 The second adjunction mapping: the case when $\kappa(\mathcal{K}) \geq 3$ and $\dim \Phi_1(X) \leq 3$

Throughout this section we assume the hypotheses and notation of Problem 1.11. We recall the notation $d_i := \mathcal{K}^i \cdot L^{n-i}$ for $i = 0, \ldots, n$ with $\mathcal{K} := K_X + (n-2)L$. We usually denote $d_0$ by $d$.

### 4.1 Lower bounds on the number of independent sections

An important result is that $\mathcal{K}$ does not have to be spanned.

**Theorem 4.1 (Lanteri, Palleschi, and Sommese [28, §1])** *Let $X$ be a smooth connected projective threefold with $-K_X = 2\mathcal{M}$ for an ample line bundle $\mathcal{M}$ and with $\mathcal{M}^3 = 1$. Then $L := 3\mathcal{M}$ is very ample, but $K_X + L$ is ample and not spanned.*

Note that $K_X + L = \mathcal{M}$ is spanned except at one point, and the image of $|K_X + L|$ is $\mathbb{P}^2$ with fibers curves of arithmetic genus 1. This is the only known example with $\mathcal{K}$ nef but not spanned. As Theorem 5.2 makes clear this example is exceptional in that $s_2$ has degree greater than one. The above result and the difficulty of showing that $\mathcal{K}$ is spanned prompted Beltrametti and this author to see what they could discover about the possibly meromorphic map $\psi_1 : X \to \mathbb{P}^{h^0(\mathcal{K})-1}$ associated to $|\mathcal{K}|$. The starting point of our investigations was an analysis of best possible lower bounds for $h^0(\mathcal{K})$. In trying to find lower bounds for $h^0(\mathcal{K})$, low degree pairs, i.e., pairs $(\hat{X}, \hat{L})$ with $\hat{L}^n$ small, cause serious difficulties. In [38] this author dealt with these cases by showing that the double point inequality

$$d(d-10) + 12\chi(\mathcal{O}_S) \geq 2d_2 + 5d_1,$$

is true for $\hat{L}^n \geq 7$, and then used this to show that $h^0(\mathcal{K}) \geq 3$ when $\kappa(\hat{X}) \geq 0$. The above inequality is a surface inequality, and it suggests trying to use a threefold double point inequality. The problem that arises with this is that the Euler characteristic of the threefold, which is not directly tied to projective invariants of the pair $(\widehat{X}, \widehat{L})$, comes into the formula. Very fortuitously, the hard Lefschetz theorem allows the formula to be transformed into the following very useful estimate.

**Theorem 4.2 (Beltrametti and Sommese [13])** *Let $(\widehat{X}, \widehat{L})$ be a smooth projective threefold, polarized with a very ample line bundle, $\widehat{L}$. Let $(X, L)$ and $\phi : \widehat{X} \to X$ be the first reduction and first reduction map, respectively. Let $\widehat{d} := \widehat{L}^3$. Let $\gamma$ be the number of points blown up by $\phi$. Let $S$ be a smooth element in $|L|$. Then*

$$44h^0(\mathcal{K}) + 60\chi(\mathcal{O}_S) + 2h^0(K_X) - 2 \geq 13d_2 + 17d_1 + d_3 + (20 - \widehat{d})\widehat{d} + 5\gamma.$$

*If $\kappa(K_{\widehat{X}} + (n-2)\widehat{L}) \geq 2$ then*

$$44h^0(\mathcal{K}) + 58\chi(\mathcal{O}_S) + 2h^0(K_X) + 4 \geq 12d_2 + 17d_1 + d_3 + (20 - \widehat{d})\widehat{d} + 5\gamma.$$

To use these inequalities, note that if $h^0(K_X) > 0$ then $h^0(\mathcal{K}) \geq h^0(L) \geq h^0(\widehat{L})$. Thus in this case we are going to get $h^0(\mathcal{K}) \geq 6$ unless $(\widehat{X}, \widehat{L})$ arises as a hypersurface in $\mathbb{P}^N$ with $N \leq 4$. Thus, if $h^0(\mathcal{K}) \leq 5$ or if $\widehat{\Phi}_1$ is not birational, we can reduce to the case when $h^0(K_X) = 0$. Combining Theorem 4.2 with Theorem 1.10 and many special adjunction theoretic arguments, the following results were deduced.

**Theorem 4.3 (Beltrametti and Sommese [13])** *Let $(\widehat{X}, \widehat{L})$ be an $n$-dimensional projective manifold polarized with a very ample line bundle, $\widehat{L}$. If $\kappa(K_{\widehat{X}} + (n-2)\widehat{L}) = 3$ then $h^0(K_{\widehat{X}} + (n-2)\widehat{L}) \geq 2$. If $\kappa(K_{\widehat{X}} + (n-3)\widehat{L}) \geq 0$ then $h^0(K_{\widehat{X}} + (n-2)\widehat{L}) \geq 5$ with equality only if $n = 3$ and $(\widehat{X}, \widehat{L})$ is a quintic hypersurface in $\mathbb{P}^4$.*

The next step was to analyze the meromorphic map $\psi_1$ associated to $|\mathcal{H}|$.

1. Is $\psi_1$ and hence also $\Phi_1$ a morphism? Note this can happen even though $\mathcal{K}$ is not spanned and even though the mapping $\psi_1 : Y \to \mathbb{P}^{h^0(\mathcal{K})-1}$ has lower dimensional image than $\kappa(\mathcal{K})$.

2. If $\dim Z_1 < 3$, what can we say about the invariants of the fibers of $\mathcal{R}_1$ and of $Z_1$?

3. What can we say about the degree of $s_1$?

We break our analysis of the second adjunction mapping up in terms of the dimension of the image of the meromorphic map $\psi_1 : Y \to \mathbb{P}^{h^0(\mathcal{K})-1}$ associated to $|\mathcal{K}|$. From Theorem 4.3 we see that this map cannot have an image of dimension 0.

### 4.2 The case when $\dim \Phi_1(X) = 1$

It is a classical result of Beauville [2] that given a smooth general type surface $S$, then if the meromorphic mapping associated to $|K_S|$ has a one-dimensional image, it 'usually' is a morphism with the genus of fibers 'small'. Beauville's theorem raised the hope that if the meromorphic map $\Phi_1$ has a one-dimensional image, then $\Phi_1$ might be a morphism. Beauville's result cannot be used in our situation because the restriction map $H^0(K_X + (n-2)L) \to H^0(K_S)$ is not onto, i.e., it does not follow from $K_S$ being spanned and $|K_S|$ having a two-dimensional image, that the map associated to $|\mathcal{K}|$ has a two-dimensional image. By an involved analysis of the numerical properties of the fibers of the meromorphic map $\mathcal{R}_1 : Y \to Z_1$, the following results are deduced in [14].

**Theorem 4.4** *Let $(\hat{X}, \hat{L})$ be a connected $n$-dimensional projective manifold polarized by a very ample line bundle $\hat{L}$. Assume that $\kappa(K_{\hat{X}} + (n-2)\hat{L}) \geq 3$. Further assume that $\Phi_1$ has a one-dimensional image. If either $h^0(\mathcal{K}) \geq 7$ or $\kappa(K_{\hat{X}} + (n-3)\hat{L}) \geq 0$ or $h^1(\mathcal{O}_{Z_1}) > 0$, then $\Phi_1$ is a morphism.*

**Theorem 4.5** *Let $(\hat{X}, \hat{L})$ be a connected $n$-dimensional projective manifold polarized by a very ample line bundle $\hat{L}$. Assume that $n \geq 3$ and $\kappa(K_{\hat{X}} + (n-3)\hat{L}) \geq 0$. Let $(X, L)$ be the first reduction of $(\hat{X}, \hat{L})$. Further assume that $\Phi_1$ has a one-dimensional image. Let $f = F \cap S$ be the transverse intersection of a general fiber $F$ of $\mathcal{R}_1$ with a general surface section $S$ of $(X, L)$. Then $\Phi_1$ is a morphism and $g(f)$, the genus of $f$, is $\leq 6$. For $h^0(\mathcal{K}) \geq 21$ we have $g(f) \leq 5$, and the intersection of $F$ with a general threefold section of $(X, L)$ is either a $K3$ surface or the blowing up at one point of a $K3$ surface, and $Z_1 := \mathcal{R}_1(Y)$ is a curve of genus $g(Z_1) \leq 1$.*

In the case when $g(f) \leq 5$ in the preceding theorem, the complete list of possible pairs, $(A \cap F, L_{A \cap F})$, where $A$ is a general threefold section of $(X, L)$, is worked out in [14]. The map $s_1$ is almost always birational.

**Theorem 4.6** *Let $(\hat{X}, \hat{L})$ be a connected $n$-dimensional projective manifold polarized by a very ample line bundle $\hat{L}$. Assume that $\kappa(K_{\hat{X}} + (n-2)\hat{L}) \geq 3$. Further assume that the meromorphic map $\Phi_1$ has a one-dimensional image. The map $s_1$ is an embedding if $h^1(\mathcal{O}_{Z_1}) = 0$. If $h^1(\mathcal{O}_{Z_1}) > 0$, then the degree of $s_1$ is at most 2. If the degree is 2, then $\chi(\mathcal{O}_S) \leq 2$ for a general surface section $S$ of $(X, L)$, $h^0(\mathcal{K}) = 2$, and $3 \leq g(f) \leq 4$, where $f = F \cap S$ is the transverse intersection of a general fiber $F$ of $\mathcal{R}_1$ with $S$.*

### 4.3 The case when $\dim \Phi_1(X) = 2$

In the case of a two-dimensional image we have been unable to show that the second adjunction mapping is usually a morphism. The following results of [15] summarize what we know.

**Theorem 4.7** *Let $(\hat{X},\hat{L})$ be a connected $n$-dimensional projective manifold polarized by a very ample line bundle $\hat{L}$. Assume that $\kappa(K_{\hat{X}}+(n-2)\hat{L}) \geq 3$. Further assume that $\Phi_1$ has a two-dimensional image. Let $\mathfrak{f}$ be a general fiber of $\Phi_1$. If $h^0(\mathcal{K}) \geq 6$, e.g., if $\kappa(K_{\hat{X}}+(n-3)\hat{L}) \geq 0$, then $L^{n-2}\cdot\mathfrak{f} \leq 13$.*

In [15] an analysis is also made for smaller values of $h := h^0(\mathcal{K})$, e.g., it is shown that if $h = 5$, then $L^{n-2}\cdot\mathfrak{f} \leq 14$; if $h = 4$, then $L^{n-2}\cdot\mathfrak{f} \leq 17$; and if $h = 3$ then $L^{n-2}\cdot\mathfrak{f} \leq 24$.

**Theorem 4.8** *Let $(\hat{X},\hat{L})$ be a connected $n$-dimensional projective manifold polarized by a very ample line bundle $\hat{L}$. Assume that $\kappa(K_{\hat{X}}+(n-2)\hat{L}) \geq 3$. Further assume that $\Phi_1$ has a two-dimensional image. Let $F$ be a general fiber of $\mathcal{R}_1$. Let $\mathcal{B}$ be the base locus of $|\mathcal{H}|$. Then $L^{n-2}\cdot F \geq 3$. Further if either $\kappa(K_{\hat{X}}+(n-3)\hat{L}) \geq 3$ or $\kappa(K_{\hat{X}}+(n-3)\hat{L}) \geq 0$ and $\dim\mathcal{B} \leq n-3$, then $L^{n-2}\cdot F \geq 4$. If $h := h^0(\mathcal{K}) \geq 6$ then the genus $g$ of a curve section of $(F,L_F)$ is $\leq 66$.*

In [15] an analysis is also made for smaller values of $h := h^0(\mathcal{K})$, e.g., it is shown that if $h = 5$ then $g \leq 78$; if $h = 4$ then $g \leq 120$; and if $h = 3$ then $g \leq 253$. Moreover if $\psi_1$ is a morphism and $h^0(\mathcal{K}) \geq 77$ then $g \leq 26$.

**Theorem 4.9** *Let $(\hat{X},\hat{L})$ be a connected $n$-dimensional projective manifold polarized by a very ample line bundle $\hat{L}$. Assume that $\kappa(K_{\hat{X}}+(n-2)\hat{L}) \geq 3$. Further assume that $\Phi_1$ has a two-dimensional image. Let $\mathcal{B}$ be the base locus of $|\mathcal{K}|$. Let $F$ be a general fiber of $\mathcal{R}_1$. Then $\deg s_1 \leq \dfrac{13}{L^{n-2}\cdot F}$ (and hence $\deg s_1 \leq 4$) if $h^0(\mathcal{K}) \geq 6$, e.g., if $\kappa(X) \geq 0$. If either $\kappa(K_{\hat{X}}+(n-3)\hat{L}) \geq 3$ or $\kappa(K_{\hat{X}}+(n-3)\hat{L}) \geq 0$ and $\dim\mathcal{B} \leq n-3$, then $\deg s_1 \leq 3$.*

### 4.4 The case when $\dim\Phi_1(X) = 3$

We summarize results from [15] on what we know when $\dim\phi_1(X) = 3$.

**Theorem 4.10** *Let $\hat{L}$ be a connected $n$-dimensional projective manifold polarized by a very ample line bundle $\hat{L}$. Assume that $n \geq 3$ and $\kappa(K_{\hat{X}}+(n-2)\hat{L}) \geq 3$. Assume further that $\Phi_1$ has a 3-dimensional image. Then $\deg s_1 \leq 53$ and if further $h^1(\mathcal{O}_X) = 0$ then $\deg s_1 \leq 31$.*

Finally we turn to higher multiples of $\mathcal{K}$. Consider the morphism $r : X \to Y$ where $Y$ is the normal projective variety of dimension 3 or $n$ with an ample line bundle $\mathcal{H}$ such that $\mathcal{K} \cong r^*\mathcal{H}$. $Y$ is very well behaved: it has at worst isolated rational singularities (they are terminal of index at most two). When $r$ is birational, $Y$ with an appropriate polarization is called the second reduction. This author worked out the structure of $Y$ in [36]. See [12] for a careful development including the higher dimensional contributions of Beltrametti and this author, Fania, and Fujita. Here is what Beltrametti and this author show in [15].

**Theorem 4.11** *Let $(\hat{X}, \hat{L})$ be a connected $n$-dimensional projective manifold $\hat{X}$ polarized by a very ample line bundle $\hat{L}$. Assume that $\kappa(K_{\hat{X}} + (n-2)\hat{L}) \geq 3$. Then $3\mathcal{H}$ is very ample.*

**Problem 4.12** *Let $(\hat{X}, \hat{L})$ be a smooth threefold polarized by a very ample line bundle $\hat{L}$. Assume that $\kappa(K_{\hat{X}} + (n-2)\hat{L}) = 3$ (and thus in particular that $2\mathcal{K}$ and $2\mathcal{H}$ are spanned). Work out the structure of the morphism $s_2$.*

The following bounds on the degree of the map $s_2$ of the last problem are taken from [16, 17]. The best general unrestricted result for threefolds is the following. We give the argument since it as a sample of the type of reasoning used.

**Theorem 4.13** *Let $(\hat{X}, \hat{L})$ be a smooth threefold polarized by a very ample line bundle $\hat{L}$. Assume that $\kappa(K_{\hat{X}} + (n-2)\hat{L}) = 3$ (and thus in particular that $2\mathcal{K}$ and $2\mathcal{H}$ is spanned). Then the morphism $s_2$ has degree at most seven.*

**Proof.** Let $(X, L)$ denote the first reduction of $(\hat{X}, \hat{L})$. Since the result is trivial for $h^0(\hat{L}) \leq 5$, we can assume that $h^0(\hat{L}) \geq 6$. Using Castelnuovo's inequality for the genus of a curve section combined with the positivity of $d, d_1, d_2, d_3$, the Hodge inequalities $dd_2 \leq d_1^2, d_1 d_3 \leq d_2^2$, and the fact that $d + d_1, d_2 + d_3$ are even, we conclude that $d \geq 8$. If $h^0(K_X + \mathcal{K}) > 0$, then $|2\mathcal{K}| = |(K_X + \mathcal{K}) + L|$ would give a birational map. Therefore we conclude that $h^0(K_X + \mathcal{K}) = \chi(K_X + \mathcal{K}) = 0$. By Riemann-Roch applied to $\chi(K_X + \mathcal{K})$ we conclude that

$$4\chi(\mathcal{O}_S) + d_2 = 6h + d_3 \tag{9}$$

where $h := h^0(\mathcal{K})$. If the theorem is false we conclude that

$$(2\mathcal{K}')^3 = 8d_3 \geq 8(h^0(2\mathcal{K}) - 3) = 8(d_2 + \chi(\mathcal{O}_S) - 3)$$

where $S$ is a smooth element of $|L|$. Combining this inequality $d_3 \geq d_2 + \chi(\mathcal{O}_S) - 3$ with equation 9, we conclude that $\chi(\mathcal{O}_S) \geq 2h - 1$. Since $h \geq 2$, this implies that $\chi(\mathcal{O}_S) \geq 3$ and therefore by the inequality $d_3 \geq d_2 + \chi(\mathcal{O}_S) - 3$ that $d_3 \geq d_2$. Since the result is trivial for $h^0(\hat{L}) \leq 5$, we can assume that $h^0(\hat{L}) \geq 6$. Using the list in [6] we conclude that if $d \leq 10$ then $h^0(\hat{L}) \geq 7$. Using this, Castelnuovo's inequality for the genus of a curve section combined with the positivity of $d, d_1, d_2, d_3$, the Hodge inequalities $dd_2 \leq d_1^2, d_1 d_3 \leq d_2^2$, and the fact that $d + d_1, d_2 + d_3$ are even, we conclude that $d \geq 10, d_1 \geq 10, d_2 \geq 10, d_3 \geq 10$. Using $\chi(\mathcal{O}_S) \geq 2h - 1$ Theorem 1.10 implies that

$$1 \geq \frac{d_1}{12} + \frac{d_3}{32}.$$

This gives the contradiction $1 \geq \dfrac{55}{48}$. Q.E.D.

## 5 The map $\hat{\Phi}_2$ when $\kappa(\mathcal{K}) \geq 3$

We continue to use the notation of Problem 1.11 We have poor knowledge about $\Phi_1$ when $\kappa(\mathcal{K}) > 3$. For $\Phi_2$ we are in much better shape, though results on degree bounds for the maps associated to $|2\mathcal{K}|$ (or $|\mathcal{K}|$) when the map has an equal dimensional image do not directly lift from threefolds to $n$-folds with $n \geq 4$. The best general unrestricted result for $n$-folds is the following.

**Theorem 5.1** *Let $(\hat{X}, \hat{L})$ be an $n$-dimensional projective manifold $\hat{X}$ polarized by a very ample line bundle $\hat{L}$. Assume that $\kappa(K_{\hat{X}} + (n-2)\hat{L}) = n \geq 3$. If either $\kappa(K_{\hat{X}} + (n-3)\hat{L}) \geq 0$ or $n \geq 6$ then $\hat{\Phi}_2$ is birational.*

We refer the reader to [17] for some weaker forms of the above result that hold in dimensions four and five, e.g., if $\kappa(K_{\hat{X}} + (n-2)\hat{L}) = n = 5$ and $\chi(\mathcal{O}_{\hat{S}}) \geq 100$ for a surface section of $(\hat{X}, \hat{L})$, then $\hat{\Phi}_2$ is birational.

**Theorem 5.2** *Let $(\hat{X}, \hat{L})$ be an $n$-dimensional projective manifold polarized by a very ample line bundle $\hat{L}$. Assume that $\kappa(K_{\hat{X}} + (n-2)\hat{L}) = n \geq 3$. If $|\mathcal{K}|$ has a smooth element, e.g., if $\mathcal{K}$ is spanned by global sections, then the map $\Phi_2$ is birational unless $n = 3$, $d_3 = 1$, $d_2 = 3$, $d_1 = 9$, and $\mathcal{H} \cong -2K_Y$. In this last case the degree is two.*

**Remark 5.3** Note that the example of Lanteri, Palleschi, and this author described in Theorem 4.1 satisfies the conditions required to be an exception to the theorem. It is in fact not hard to show that in this case the map associated to $|2\mathcal{K}|$ has degree two. This is the only known example where $\mathcal{K}$ is nef and big and $s_2$ is not birational. Since we expect $\mathcal{K}$ to usually be spanned, there is the obvious hope that this is the only example where $s_2$ is not birational.

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