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ASPECTS OF OPERATOR THEORY IN HEREDITARILY INDECOMPOSABLE BANACH SPACES

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ABSTRACT. We examine certain special features exhibited by various classes of linear operators acting in a hereditarily indecomposable Banach space. For instance, we show that the family of all Riesz operators in a H.I. space forms a closed, 2-sided ideal. We also give further characterizations of the class of scalar-type spectral operators (to those already given in [16]). The final section discusses some properties of the spectral maximal spaces of (necessarily decomposable) linear operators in such spaces.

1 Introduction

A Banach space X is called *hereditarily indecomposable* (briefly, H.I.) if, whenever Y and Z are closed, infinite dimensional subspaces of X and $\delta > 0$, then there exist unit vectors $y \in Y$ and $z \in Z$ such that

$$\|y - z\| < \delta.$$

This is equivalent to the following property: whenever Y and Z are closed, infinite dimensional subspaces of X satisfying $Y \cap Z = \{0\}$, then $Y + Z$ is non-closed. Examples of H.I. spaces were first exhibited by Gowers and Maurey, [9]. Since then, this class of Banach spaces has been intensively studied and has had an important influence on the geometry of Banach spaces; see [5], [6], [7], [8], [9], [10], and the references therein. The space of linear operators in H.I. spaces is rather special; given any bounded linear operator T there is a unique point λ_T in the spectrum $\sigma(T)$ of T such

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that $T - \lambda_T I$ is strictly singular, [9]. See also [5], [14], [15], [16], for further properties of certain classes of linear operators in H.I. spaces. The aim of this note is to continue with the investigation begun in [14], [15], [16] concerning certain aspects of operator theory in H.I. spaces. It is known that the collection of all Riesz operators (even in a Hilbert space) is in general neither a vector space, nor an ideal, nor a closed set for the operator norm topology. In Section 2 we show that in any H.I. space the family of all Riesz operators forms an operator norm closed, 2-sided ideal. Section 3 is concerned with various characterizations of the class of scalar-type spectral operators in a H.I. space, thereby extending [16; Proposition 2]. It is known that the infinitesimal generator of a C_0-group of linear operators in a H.I. space is always bounded [14]; this is not the case in general for the generator A of a C_0-semigroup, [14, Example 2.4]. Various sufficient conditions are presented in [15] which guarantee that A is bounded. In Section 4 we present two further results in this direction.

2 Riesz operators

Given a Banach space X let $L(X)$ denote the space of all continuous linear operators of X into itself. The dual space of X is denoted by X'. If X is a H.I. space and $T \in L(X)$, than there exists a unique point $\lambda_T \in \mathbb{C}$ such that $T - \lambda_T I$ is strictly singular. Moreover, $T - \lambda_T I$ is a Riesz operator (defined below) and λ_T is the unique point of $\mathbb{C}$ with this property. In particular, $\sigma(T)$ is a finite set or consists of a sequence of eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ converging to λ_T. Moreover, every element of $\sigma(T)\setminus\{\lambda_T\}$ is an isolated point of $\sigma(T)$ and has an associated spectral projection of finite rank; see [9], [14]. Let X be a Banach space. The closed, 2-sided ideal in $L(X)$ consisting of the compact operators is denoted by $K(X)$. For $T \in L(X)$ define

$$\kappa(T) = \inf\{\|T - A\| : A \in K(X)\}.$$

Since $K(X)$ is operator norm closed in $L(X)$ it is clear that $\kappa(T) = 0$ iff $T \in K(X)$. An operator $T \in L(X)$ is called a *Riesz operator* if $\lim_{n\to\infty}[\kappa(T^n)]^{1/n} = 0$; for the general theory of such operators we refer to [4], for example. Let $R(X)$ denote the collection of all Riesz operators. Clearly $K(X) \subseteq R(X)$. At this stage it is instructive to consider an example in order to see how Riesz operators behave.

Example 2.1 ([4; Example 3.6]). Let $X = l^2$. For each $x = (x_1, x_2, \dots)$ in X define $Sx = (0, x_1, 0, x_3, \dots)$ and $Tx = (x_2, 0, x_4, 0, \dots)$,in which case both $S, T \in L(X)$. Since $S^2 = 0 = T^2 \in K(X)$ it is clear that both S, T are Riesz operators. However, since $STx = (0, x_2, 0, x_4, \dots)$ for each $x \in X$ we see that ST is a continuous projection of infinite rank and so $ST \notin K(X)$. But,

$(ST)^n = ST$ for all $n \geq 1$ implies that $\kappa((ST)^n) = \kappa(ST) = \alpha > 0$ for all $n \geq 1$, and so $lim_{n\to\infty}[\kappa((ST)^n)]^{1/n} = 1$. Accordingly, $ST \notin R(X)$. Since $S \notin K(X)$(otherwise $ST \in K(X)$) we see that $K(X)$ is a proper subset of $R(X)$. Noting that $(S+T)^2 = I$ it follows that $(S+T)^{2n} = I$ for all $n \geq 1$ and so $lim_{n\to\infty}[\kappa((S+T)^{2n})]^{1/2n} = 1$. Accordingly, $(S+T) \notin R(X)$. These calculations show that $R(X)$ is neither an ideal nor a vector space in $L(X)$. Moreover, $R(X)$ also fails to be operator norm closed in $L(X)$, [4; Example 3.15].

In view of the above example the following result is somewhat surprising.

Proposition 2.2 *Let X be a H.I. space. Then the family of all Riesz operators $R(X)$ forms an operator norm closed, 2-sided ideal in $L(X)$ of co-dimension one.*

Proof. Let $T \in L(X)$. If T is strictly singular, then T is a Riesz operator [13; 26.6.5]. Conversely, if T is a Riesz operator, $T - \lambda_T I$ is strictly singular (by [9; § 4] this is true for *all* $T \in L(X)$) and hence a Riesz operator. By [14; Proposition 1.1] λ_T is the unique element μ of $\mathbb{C}$ such that $T - \mu I$ is a Riesz operator. Thus $\lambda_T = 0$, i.e. T is strictly singular. Accordingly, $R(X)$ coincides with the set of all strictly singular operators $S(X)$ which is a 2-sided ideal, [13; 1.9.4], and has co-dimension one, [9; § 4]. Let $\phi : L(X) \to \mathbb{C}$ be defined by $\phi(T) = \lambda_T$. It follows easily from

$$(\alpha S + \beta T) - (\alpha\lambda_S + \beta\lambda_T)I = \alpha(S - \lambda_S I) + \beta(T - \lambda_T I) \in S(X),$$

$$TS - \lambda_T\lambda_S I = T(S - \lambda_S I) + \lambda_S(T - \lambda_T I) \in S(X), \text{ and}$$

$$ST - \lambda_S\lambda_T I = S(T - \lambda_T I) + \lambda_T(S - \lambda_S I) \in S(X),$$

that ϕ is a linear algebraic homomorphism. Moreover, $\phi \neq 0$ since $\phi(I) = 1$. Since $L(X)$ is a unital Banach algebra it follows from standard Banach algebra theory that ϕ is continuous. Thus

$$\ker(\phi) = S(X) = R(X)$$

is an operator norm closed, 2-sided ideal in $L(X)$ of co-dimension one. □

It is interesting to re-examine Example 2.1 in the setting of H.I. spaces with a Schauder basis. So, for $x \in X$ we write $x = (x_1, x_2, \dots)$ with respect to this basis. Define linear subspaces

$$D(S) = \{x \in X : (0, x_1, 0, x_3, \dots) \in X\}$$

and

$$D(T) = \{x \in X : (x_2, 0, x_4, 0, \dots) \in X\}$$

and linear operators $S:D(S) \to X$ and $T:D(T) \to X$ by $Sx = (0, x_1, 0, x_3, \dots)$ and $Tx = (x_2, 0, x_4, 0, \dots)$, respectively. Since every $x \in X$ with finite support belongs to both $D(S)$ and $D(T)$ it is clear that $D(S)$ and $D(T)$ are both dense in X. Using the continuity of the co-ordinate functionals in X it is routine to verify that S and T are both closed operators. Moreover, $S(D(S)) \subseteq D(S)$ with $S^2x = 0$ for all $x \in D(S)$ and $T(D(T)) \subseteq D(T)$ with $T^2x = 0$ for all $x \in D(T)$. By definition of the composition of unbounded operators we have $D(ST) = \{x \in D(T) : Tx \in D(S)\}$ and it is easily calculated that

$$(ST)x = (0, x_2, 0, x_4, \dots), \qquad x \in D(ST).$$

Hence, ST is a densely defined linear operator in X. The difference between ST here and that in Example 2.1 is that now ST is *unbounded* on $D(ST)$. To see this, assume the contrary. Then ST would have an extension, say $\Lambda \in L(X)$, given by $\Lambda x = (0, x_2, 0, x_4, \dots)$ for all $x \in X$. But, $\Lambda^2 = \Lambda$ and Λ has infinite dimensional range. Then $I - \Lambda : x \mapsto (x_1, 0, x_3, 0, \dots)$ is also a continuous projection with infinite dimensional range. Since X is a H.I. space and $X = \Lambda X \oplus (I - \Lambda)X$ this is possible.

3 Scalar-type spectral operators

Recall that a bounded linear operator in a Banach space X is a *scalar-type spectral operator* if there exists a spectral measure $E : \mathcal{B}(\sigma(T)) \to L(X)$, defined on the Borel subsets $\mathcal{B}(\sigma(T))$ of $\sigma(T)$, which is σ-additive for the strong operator topology and satisfies $T = \int_{\sigma(T)} z\, dE(z)$; see [4] for the general theory of such operators. Suppose that $\mathcal{F}$ is a Banach algebra of $\mathbb{C}$-valued functions on some set $\Omega \subseteq \mathbb{C}$ such that $e_n : z \mapsto z^n$, for $z \in \Omega$, belongs to $\mathcal{F}$ for each integer $n \geq 0$. Then an operator $T \in L(X)$ is said to admit an *$\mathcal{F}$-functional calculus* if there is a Banach algebra homomorphism $\phi : \mathcal{F} \to L(X)$ such that $\phi(e_0) = I$ and $\phi(e_1) = T$. The Banach algebra of all bounded Borel functions on Ω is denoted by $B^\infty(\Omega)$; it is equipped with the sup-norm $\|\cdot\|_\infty$. The closed subalgebra of all bounded continuous functions is denoted by $C(\Omega)$. The following result is an extension of [16; Proposition 2].

Proposition 3.1 *Let X be a H.I. space and $T \in L(X)$. Then the following statements are equivalent.*
(i) T is a scalar-type spectral operator.
(ii) There exist finitely many non-zero, pairwise disjoint projections $P_1, \dots, P_n$, with exactly one having infinite rank and satisfying $\sum_{j=1}^n P_j = I$, such that $T \in$ span $\{P_1, \dots, P_n\}$.
(iii) T admits a $C(\sigma(T))$-functional calculus.
(iv) T admits a $B^\infty(\sigma(T))$-functional calculus.

(v) There exists $K > 0$ *such that, for each complex polynomial* p,

$$\| p(T) \| \le K \sup\{| p(\lambda) | : \lambda \in \sigma(T)\}.$$

Proof. Since H.I. spaces cannot contain a copy of the sequence space c_0, the equivalence $(i) \Leftrightarrow (iii)$ follows from [3; Theorem 3.1]. Similarly, since H.I. spaces cannot contain a copy of l^∞, the equivalence $(i) \Leftrightarrow (iv)$ follows from [3; Theorem 3.3]. The equivalence $(i) \Leftrightarrow (ii)$ is part of [16; Proposition 2].

$(iii) \Leftrightarrow (v)$. The direction $(iii) \Rightarrow (v)$ is clear. So, assume that (v) holds. Since $\sigma(T)$ is a countable set with at most one limit point, [14; Proposition 1.1], it follows from Mergelyans's theorem, [17; p. 423], that the polynomials restricted to $\sigma(T)$ are dense in $C(\sigma(T))$. So, given $f \in C(\sigma(T))$ there exist polynomials $\{p_n\}_{n=1}^\infty$ such that $p_n \to f$ uniformly on $\sigma(T)$. Then

$$\|p_n - p_m\|_\infty \le \|p_n(T) - p_m(T)\| \le K\|p_n - p_m\|_\infty, \qquad m, n \in \mathbb{N},$$

where the first inequality follows from the spectral mapping theorem and the fact that the spectral radius of an operator is dominated by the norm of the operator. So, there exists $\phi(f) \in L(X)$ such that $p_n(T) \to \phi(f)$ in operator norm. Since $p \mapsto \phi(p) := p(T)$ is linear, multiplicative and continuous, so is its extension $f \mapsto \phi(f)$ to $C(\sigma(T))$. □

It follows from Proposition 3.1 that any scalar-type spectral operator in a H.I. space X has finite spectrum. In particular, if $T \in L(X)$ has infinite spectrum, then there must exist a sequence of polynomials $\{p_n\}_{n=1}^\infty$ with

$$\|p_n\|_\infty := \sup\{|p(\lambda)| : \lambda \in \sigma(T)\} \le 1, \qquad n \in \mathbb{N},$$

such that $\sup_{n\in\mathbb{N}}\|p_n(T)| = \infty$. So, if $Z(T)$ denotes the space of all vectors $x \in X$ for which

$$|||x||| = \sup\{\|p(T)x\| : p \text{ a polynomial}, \|p\|_\infty \le 1\} < \infty,$$

(cf. [3; p.165), then it follows that $Z(T) \neq X$. For an operator $T \in L(X)$ with $\sigma(T) \subseteq \mathbb{R}$, R. deLaubenfels defines

$$\|x\|_{Z(T)} = \sup\{\| \prod_{k=1}^{m} \lambda_k^{n_k}(\lambda_k I - iT^k)^{-n_k} x\| : \lambda_k \in \mathbb{R}\backslash\{0\} \text{ and } (m-1), n_k \in \mathbb{N}\},$$

in which case it turns out that $Z(T) = \{x \in X : \|x\|_{Z(T)} < \infty\}$ and that $\|\cdot\|_{Z(T)}$ and $|||\cdot|||$ are equivalent norms on $Z(T)$, [3; p.165]. Moreover, $(Z(T), \|\cdot\|_{Z(T)})$ is always a Banach space which is continuously imbedded in X; it is called the *semi-simplicity manifold* of T and was originally introduced by Sh. Kantorovitz. For the notion of a *well bounded operator* $T \in L(X)$ we refer to [4]. For such an operator there exists a family $\{E(\lambda)\}_{\lambda\in\mathbb{R}}$ of projections in $L(X')$, called a *decomposition of the identity* for T (with certain properties; see [1; Theorem 3.2]), such that

$$\langle Tx, x'\rangle = b\langle x, x'\rangle - \int_a^b \langle x, E(\lambda)x'\rangle d\lambda, \quad x \in X, x' \in X'.$$

Here the compact interval $[a,b]$ has the property that $\sigma(T) \subseteq [a,b]$, [4; Corollary 15.9]. If the function $\lambda \mapsto \langle x, E(\lambda)x'\rangle$, for $\lambda \in \mathbb{R}$, is of bounded variation for each $x \in X$ and $x' \in X'$, then the decomposition $\{E(\lambda)\}_{\lambda\in\mathbb{R}}$ is said to be of *bounded variation.* For scalar-type spectral operators with *real spectrum* Proposition 3.1 has a further extension.

Proposition 3.2 *Let X be a H.I. space and $T \in L(X)$ satisfy $\sigma(T) \subseteq \mathbb{R}$. Then the following statements are equivalent.*

(i) T is a scalar-type spectral operator.

(ii) $Z(T) = X$.

(iii) T is well bounded with a decomposition of the identity of bounded variation.

Proof. $(i) \Rightarrow (ii)$. By Proposition 3.1 the operator T has a $C(\sigma(T))$-functional calculus, say ϕ, and so

$$K = \sup\{\|\phi(f)\| : f \in C(\sigma(T)), \|f\|_\infty \le 1\} < \infty.$$

In particular, $|||x||| < \infty$ for all $x \in X$ and so by earlier remarks $Z(T) = X$.

$(ii) \Rightarrow (i)$. The identity function $\Lambda : (X, |||\cdot|||) \to X$ is continuous (just put $p = e_0$ in the definition of $|||\cdot|||$ to see that $\|x\| \le |||x|||$ for all $x \in X$). By the open mapping theorem Λ^{-1} is continuous and so

$$\begin{aligned} &\sup\{\|p(T)\| : p \text{ a polynomial}, \|p\|_\infty \le 1\} \\ &\quad = \sup_{\|y\|\le 1} \sup_{\|p\|_\infty \le 1} \|p(T)y\| = \sup_{\|y\|\le 1} |||y||| = \|\Lambda^{-1}\| \end{aligned}$$

is finite. Then Proposition 3.1 shows that T is a scalar-type spectral operator.

$(i) \Rightarrow (iii)$. By [1; Theorem 5.4] the operator T is well bounded of type (B) and hence, in particular, has a *unique* decomposition of the identity. Proposition 3.1 implies that $T = \Sigma_{j=1}^n z_j P_j$ for some set $\{P_j\}_{j=1}^n$ of non-zero, paiwise disjoint projections with $\Sigma_{j=1}^n P_j = I$, where $\sigma(T) = \{z_j\}_{j=1}^n \subseteq \mathbb{R}$. Then the (unique) decomposition of the identity $\{E(\lambda)\}_{\lambda\in\mathbb{R}} \subseteq L(X')$ of T is given by $E(\lambda) = \sum_{z_j \le \lambda} P_j'$, for each $\lambda \in \mathbb{R}$. It is then clear that

$$\lambda \mapsto \langle x, E(\lambda)x'\rangle = \sum_{z_j \le \lambda} \langle P_j x, x'\rangle, \lambda \in \mathbb{R},$$

is of bounded variation for each $x \in X$ and $x' \in X'$.

$(iii) \Rightarrow (i)$. By [1; Theorem 5.2], with $J \supseteq \sigma(T)$, there is $K > 0$ such that $\|p(T)\| \le K\|p\|_\infty$ for every complex polynomial p. Then Proposition 3.1 shows that T is a scalar-type spectral operator. □

4 C_0-semigroups

As mentioned before C_0-*semigroups* in H.I. spaces (unlike C_0-groups) do not necessarily have bounded infinitesimal generators. In [15] we presented some special classes of C_0-semigroups which do have bounded infinitesimal generators in a H.I. space. We present here two further results in this direction. The domain of an unbounded linear operator A is denoted by $D(A)$. Let $\Gamma = \{z \in \mathbb{C} : |z| = 1\}$. For the general theory of C_0-semigroups we refer to [12].

Proposition 4.1 *Let X be a H.I. space and $(T(t))_{t\geq 0}$ to be a C_0-semigroup in X. If there exist $t_1, t_2 \in (0,\infty)$ which are rationally independent and such that $T(t_j)$ are scalar-type spectral operators with $\sigma(T_j) \subseteq \Gamma$, for each $j \in \{1,2\}$, then the infinitesimal generator $A \in L(X)$ and $A - \lambda_A I$ is a finite rank operator.*

Proof. By [11; Theorem 4] the semigroup $(T(t))_{t\geq 0}$ has an extension to a C_0-group on $\mathbb{R}$ and its generator iA (hence A) is a scalar-type spectral operator. Since no unbounded scalar-type spectral operators exist in H.I spaces, [16; Proposition 2(v)], it follows that $A \in L(X)$. By Proposition 3.1 we have $A = \lambda_0 E_0 + \lambda_1 E_1 + \ldots + \lambda_n E_n$; here $\lambda_0 = \lambda_A$ with $\sigma(A) = \{\lambda_j\}_{j=0}^n$ and $\{E_j\}_{j=0}^n$ generates the resolution of the identity of A with $\dim(E_0 X) = \infty$ and dim $(E_j X) < \infty$ for all $1 \leq j \leq n$. Since $A - \lambda_A I = \sum_{j=1}^n (\lambda_j - \lambda_0) E_j$ it is clear that $A - \lambda_A I$ is a finite rank operator. □

Proposition 4.2 *Let X be a H.I. space and $(T(t))_{t\geq 0}$ be a C_0-semigroup in X such that its infinitesimal generator A satisfies $(0,\infty) \subseteq \rho(A)$ and $T(t)X \subseteq D(A)$, for all $t > 0$. If*

$$\sup\{\frac{1}{(k-1)!}\int_0^\infty t^{k-1}|\langle A^k T(t)x, x'\rangle \mid dt : k \in \mathbb{N}, \|x\| \leq 1, \|x'\| \leq 1\}$$

is finite, then $A \in L(X)$ and $A - \lambda_A I$ is a finite rank operator.

Proof. Since X cannot contain a copy of the sequence space c_0 it follows that $-A$ is a scalar-type spectral operator, [19; Theorem 1]. But, as noted above, unbounded scalar-type spectral operators do not exist in H.I. spaces and so $A \in L(X)$. Arguing as in the proof of Proposition 4.1 it follows that $A - \lambda_A I$ is a finite rank operator. □

We recall the fact if $(T(t))_{t\geq 0}$ is a C_0-semigroup in a H.I. space X consisting of scalar-type spectral operators, then its infinitesimal generator A satisfies $A \in L(X)$; see [15; Proposition 2.4]. The class of well bounded operators of type (B) is often interpreted as a natural extension of the class of scalar-type spectral operators with real spectrum. Indeed, a well bounded operator S of type (B) also has a certain spectral decomposition of the kind $S = \int_a^b \lambda\, dE(\lambda)$, where $[a,b] \subseteq \mathbb{R}$ is an interval containing $\sigma(S)$ and $E : \mathbb{R} \to L(X)$ is a certain projection valued function (usually called a spectral family). However the "integral" $\int_a^b \lambda\, dE(\lambda)$ involved is weaker than that for scalar-type spectral operators; see [4]. So, there arises quite naturally the question of whether the generator A of a C_0-semigroup $(T(t))_{t\geq 0}$ in a H.I. space consisting of well bounded operators of type (B) also satisfies $A \in L(X)$? It turns out that this need *not* be the case, in general. Indeed, let $\{e_n\}_{n=1}^{\infty}$ be a basic sequence in a H.I. space Z and let X be the closed linear span of $\{e_n\}_{n=1}^{\infty}$. Then X is also a H.I. space. Define A on X by $A(\sum_{n=1}^{\infty} \alpha_n e_n) = \sum_{n=1}^{\infty}(-n\alpha_n)e_n$ with domain $D(A) = \{\sum_{n=1}^{\infty} \alpha_n e_n \in X : \sum_{n=1}^{\infty}(-n\alpha_n)e_n \in X\}$. It is shown in [14; Example 2.4] that A generates a C_0-semigroup $(T(t))_{t\geq 0} \subseteq L(X)$ given by

$$T(t)(\sum_{n=1}^{\infty} \alpha_n e_n) = \sum_{n=1}^{\infty} e^{-nt}\alpha_n e_n, \quad \sum_{n=1}^{\infty} \alpha_n e_n \in X.$$

Since $\lambda_n(t) := -e^{-nt}$, for $n \in \mathbb{N}$, is an infinite sequence of distinct numbers in $[-1,0]$ satisfying $\lambda_n(t) \uparrow 0$, for each $t > 0$, it follows from the proof of Theorem 3 in [16] that each operator $-T(t)$, for $t \geq 0$, is well bounded of type (B). Accordingly, $(T(t))_{t\geq 0}$ is a C_0-semigroup consisting of well bounded operators of type (B), but its generator A is unbounded.

5 Spectral maximal spaces

Every bounded linear operator T in a H.I. space is necessarily a *decomposable operator*, [14; Proposition 1.1] in the sense of C. Foias; for the general theory of such operators we refer to [2], [18]. In particular, T has the *single-valued extension property* and $\sigma(T)$ coincides with the approximate point spectrum of T. Since every point $\lambda \in \sigma(T)\backslash\{\lambda_T\}$ is an isolated point of $\sigma(T)$ it follows that the *spectral maximal space* $X_T(\{\lambda\})$ associated to the closed set $\{\lambda\}$ (cf. [2] for the definition) is precisely the range $E_\lambda X$ of the spectral projection E_λ associated to the spectral set $\{\lambda\}$. In particular, $X_T(\{\lambda\}) = E_\lambda X$ is a finite dimensional subspace of X. It is also possible to give a description of the spectral maximal space $X_T(\{\lambda_T\})$. Indeed, since $T - \lambda_T I$ is also decomposable it follows from Lemma 4.4 on p.113 of [2] that

$$X_{T-\lambda_T I}(\{0\}) = \{x \in X : \lim_{n\to\infty} \|(T - \lambda_T I)^n x\|^{1/n} = 0\}.$$

Then Theorem 1.6 on p.7 of [2] implies that

$$X_T(\{\lambda_T\}) = \{x \in X : \lim_{n\to\infty} \|(T - \lambda_T I)^n x\|^{1/n} = 0\}.$$

By Definition 3.1 on p.18 and Theorem 1.5 on p.31 of [2] we note that $X_T(\{\lambda_T\})$ is a closed subspace of X. Since $X_T(\{\lambda_T\})$ is invariant for T (hence, also for $T - \lambda_T I$) it follows that the restriction $(T - \lambda_T I)|_{X_T(\{\lambda_T\})}$ is *quasinilpotent* in $L(X_T(\{\lambda_T\}))$; see the Lemma on p.28 of [2].

It is clear that $\ker(T - \lambda_T I) \subseteq X_T(\{\lambda_T\})$. Typically this inclusion is strict. Indeed, if $V = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is considered as an operator in any 2-dimensional subspace Y of X, then $V^2 = 0$ in $L(Y)$. Write $X = X_1 \oplus Y$ in which case X_1 is again a H.I. space. Then $T = I_{X_1} \oplus V$ is an element of $L(X)$ and satisfies $\lambda_T = 1$ and $\ker(T - \lambda_T I) = X_1$, which is a *proper* subspace of $X_T(\{\lambda_T\}) = X$. By replacing Y with any m-dimensional subspace of X and letting $V \in L(Y)$ be any nilpotent operator of order $k \in \{1, 2, \dots, m\}$ it is clear that there exist nilpotent operators in $L(X)$ of any given finite order. Of course, since X is infinite dimensional it must also contain quasinilpotent operators which are not nilpotent. Indeed, as pointed out by E. Albrecht it is possible (via the Hahn-Banach Theorem) to construct commuting, nilpotent, finite rank operators $T_n \in L(X)$ such that $T_n^n \neq 0$, for each $n \in \mathbb{N}$. Let $\mathcal{A}$ denote the radical, commutative Banach algebra in $L(X)$ generated by $\{T_n\}_{n=1}^{\infty}$. If all quasinilpotent operators in X are already nilpotent, then all elements of $\mathcal{A}$ are nilpotent of some finite order. By a Baire category argument it follows that all the elements of $\mathcal{A}$ must have order less than or equal to some fixed integer $m \in \mathbb{N}$. This contradicts the fact that $\mathcal{A}$ contains elements of arbitrarily high order.

So, let $T \in L(X)$ be a quasinilpotent operator which is not nilpotent. Then $\sigma(T) = 0$, that is $\lambda_T = 0$, and $X_T(\{\lambda_T\}) = X$. Moreover, $T - \lambda_T I = T$ and so the basic conjecture in H.I. spaces, which is that every operator $S \in L(X)$ has the property that $S - \lambda_S I$ is compact, gives rise to the (possibly more tractable) question of whether every quasinilpotent operator is compact?

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