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# A ONE DIMENSIONAL BOLTZMANN EQUATION WITH INELASTIC COLLISIONS

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ABSTRACT. We consider the Boltzmann equation for inelastic particles on the line and prove some preliminary results on existence and uniqueness of the solutions. We also discuss some connections with another kinetic equation investigated by the same authors.

## 1 The model

In recent times particle systems interacting via inelastic collisions have provoked an increasing interest due to the fact that they constitute a simple mathematical model for granular media, see e.g. Ref.s [5,7,10,11,12,13,14, 15,17] for preliminary physical considerations on the behavior of such systems. Unfortunately very few rigorous results are known. In this paper we attempt a mathematical study in the simple one dimensional case.

Let us consider a system of $N$ particles in $\mathbb{R}$. Let $x_i, v_i \in \mathbb{R}$ be the position and the velocity of the i-th particle and

$$Z^N = (X^N, V^N) = (x_1, v_1, ... x_N, v_N)$$

a state of the system. The dynamics is the following. The particles move freely up to the first instant in which two of them are in the same point. They collide according to the following rule:

$$v' = v - \varepsilon(v - v_1)$$

$$v_1' = v_1 + \varepsilon(v - v_1),$$

$v', v_1'$ and $v, v_1$ are the outgoing and ingoing velocity respectively, and $\varepsilon \in [0, 1/2]$ is a parameter measuring the degree of inelasticity of the collision.

Note that the collision preserves the total momentum and dissipates the kinetic energy. Moreover for $\varepsilon = 0$ we have the free particle system, while for $\varepsilon = 1/2$ we have the so called sticky particle model in which the particle pair remains attached after the collision.

A relevant qualitative feature of the systems is the possibility of delivering collapses in a finite time (for a suitable values of $\varepsilon$). Indeed it can be shown that, if $\lambda = \varepsilon N$ is sufficiently large, all the particles can reach the same position and have the same momentum after a finite time and an infinite number of collisions. See Ref. [9] for the case $N = 3$ and Ref.[2] for general $N$.

The dynamics of the system is certainly complex so that, in analogy with the standard theory of rarefied gases, it is natural to derive a reduced description given in terms of a Boltzmann equation. Obviously such a description will have a limited range of validity but, for the moment, we shall disregard this fundamental aspect.

Standard arguments of kinetic theory will lead us to consider the following equation for the unknown $f = f(x, v, t)$ that is the probability density of a single particle:

$$\partial_t f(x,v,t) + \partial_x f(x,v,t) = \\ = l \int dv_1 |v - v_1| \left( \frac{f(x,v*,t) f(x,v_1^*,t)}{(1-2\varepsilon)^2} - f(x,v,t) f(x,v_1,t) \right), \quad (1)$$

where $v^* = v + \frac{\varepsilon}{1-2\varepsilon}(v - v_1), v_1^* = v_1 - \frac{\varepsilon}{1-2\varepsilon}(v - v_1)$, are the pre-collisional velocity and $l > 0$ is the mean free time inverse.

How to justify the introduction of this equation on the basis of logically well founded arguments? One can say that Eq. (1) is a simplified model of the more difficult two and three dimensional Boltzmann equation for rarefied gas of inelastic balls in the so called Boltzmann-Grad limit (see e.g. Ref. [8]). One the other hand, as we shall see in the following, Eq. (1) can be directly derived in terms of a stochastic systems of inelastic particles.

We also note that in Ref.[4] we have obtained another kinetic equation describing the particle system in a mean field limit. This equation read as:

$$(\partial_t + v\partial_x) f(x,v,t) = -\lambda \partial_v (Ff)(x,v,t), \quad (2)$$

where:

$$F(x,v,t) = \int d\overline{v} (\overline{v} - v) |\overline{v} - v| f(x, \overline{v}, t). \quad (3)$$

It is not difficult to show, formally, that Eq. (1) tends to Eq.s (2) - (3) in the limit $\varepsilon \longrightarrow 0, l \longrightarrow \infty, l\varepsilon \rightarrow \lambda$

We notice that the homogeneous Eq. (2) with a Fokker-Plank term simulating a reservoir at a constant temperature, has also been studied in a forthcoming paper [3]. The most remarkable fact is that the asymptotic in time can be carried out rigorously and that the unique invariant state is not Maxwellian. This shows that granular media exhibit an anomalous thermodynamical behavior. We also mention that in Ref. [16] the authors propose a numerical study of the thermodynamical behavior of the particle system in a thermal reservoir.

The plan of the paper is the following. In Sect. 2 we formally derive Eq.s (1) and Eq. (2-3) under suitable scaling limits.

In Sect. 3 we deal with the simple homogeneous case and prove that the solutions to the initial value problem associated to Eq. (1) converge to the corresponding (i.e. with the same initial datum) solutions to problem (2-3) when $\varepsilon \longrightarrow 0, l \longrightarrow \infty, l\varepsilon \longrightarrow \lambda$.

In Sect. 4 we tackle the initial value problem for Eq. (1) for general $L_1$ data and show existence and uniqueness of the solutions for small $l$. We also discuss the difficulty of dealing with a large $l$ and show that a total collapse cannot occur in a finite time. However we cannot exclude other kind of singularities.

## 2 Formal derivation of the kinetic equations from particle systems

The ordinary differential equation governing the time evolution of the particle system introduced in Sect. 1 is:

$$\dot{x}_i = v_i, \qquad \dot{v}_i = \varepsilon \sum_{j=1}^{N} \delta(x_i - x_j)(v_j - v_i)|v_j - v_i|. \tag{4}$$

Notice that $\varepsilon(v_j - v_i)$ is the jump performed by the particle $i$ after a collision with the particle $j$, while $\delta(x_i - x_j)|v_j - v_i| = \delta(t - t_i, j)$, being $t_i, j$ the instant of the impact between the particle $i$ with the particle $j$.

Let $\mu^N = \mu^N(x_1, v_1, \dots, x_N, v_N)$ be a probability density for the system. The Liouville equation describing its time evolution is:

$$(\partial_t + \sum_{i=1}^{N} v_i \partial_{x_i})\mu^N(x_1, v_1, \dots, x_N, v_N) =$$

$$-\varepsilon \sum_{i \neq j} \delta(x_i - x_j)\partial_{v_i}\mu[\phi(x_1, v_1, \dots, x_N, v_N)] \tag{5}$$

where $\phi(\overline{v} - v) = (\overline{v} - v)|\overline{v} - v|$.

Proceeding as in the derivation of the BBKGY hierarchy for Hamiltonian systems, we introduce the $j$-particle distribution functions:

$$f_j^N(x_1, v_1, \ldots, x_j, v_j) = \int dx_{j+1} dv_{j+1} \ldots dx_N dv_N \mu^N(x_1, v_1, \ldots, x_N, v_N) \tag{6}$$

and integrating over the last variables Eq. (2), we obtain the following hierarchy of equations:

$$(\partial_t + \sum_{i=1}^{j} v_i \partial_{x_i}) f_j^N(x_1, v_1, \ldots, x_j, v_j) =$$

$$-\varepsilon \sum_{i \neq k}^{j} \delta(x_i - x_k) \partial_{v_i} [\phi(v_k - v_i) f_j^N(x_1, v_1, \ldots, x_j, v_j)] +$$

$$-\varepsilon(N - j) \sum_{i=1}^{j} \partial_{v_i} \int dv_j + 1 \phi(v_{j+1} - v_i) f_{j+1}^N(x_1, v_1, \ldots, x_i, v_{j+1}) \tag{7}$$

An inspection of Eq. (7) suggest the scaling limit $\varepsilon \longrightarrow 0, N \longrightarrow \infty$ in such a way that $N\varepsilon \longrightarrow \lambda$, where $\lambda$ is a positive parameter. If $f_j^N$ have a limit (say $f_j$) they are expected to satisfy the following (infinite) hierarchy of equations:

$$(\partial_t + \sum_{i=1}^{N} v_i \partial_{x_i}) f_j(x_1, v_1, \ldots, x_j, v_j) =$$

$$-\lambda \sum_{i=1}^{j} \partial_{v_i} \int dv_{j+1} \phi(v_{j+1} - v_i) f_{j+1}(x_1, v_1, \ldots, x_i, v_{j+1}). \tag{8}$$

Finally, if the initial state is chaotic, namely of initially:

$$f_j(x_1, v_1, \ldots, x_j, v_j) = \prod_{i=1}^{j} f_0(x_i, v_i, t),$$

then we expect that the limiting dynamics does not creates correlations (propagation of chaos) so that:

$$f_j(x_1, v_1, \ldots, x_j, v_j; t) = \prod_{i=1}^{j} f(x_i, v_i, t)$$

by which we obtain, for the one particle distribution function, the kinetic equation:

$$(\partial_t + v\partial_x)f(x,v) = -\lambda\partial_v(Ff), \tag{9}$$

where $F$ is given by Eq. (3) In facts products of solutions of Eq. (2) are solutions of the hierarchy (7) as follows by a simple algebraic computation.

Consider now the stochastic particle systems defined in the following way. The particle move freely up to the first impact time. Then they collide with the usual rule with probability $\alpha$ and go ahead with probability $1 \longrightarrow \alpha$.

Suppose now that the $N$-particles system is described at time 0 by a symmetric probability density $\mu_0^N$ defined on $\mathbb{R}^{2N}$. The probability density at time $t$, denoted by $\mu^N(Z^N, t)$, solves the following linear differential equations:

$$\partial_t\mu^N(Z^N,t) + \sum_{i=1}^{N} v_i\partial_{x_i}\mu^N(Z^N,t) =$$

$$\alpha\sum_{i\neq j}\delta(x_i - x_j)|v_i - v_j|(\mu^N(Z'_N(i,j)) - \mu^N(Z_N)), \tag{10}$$

where $Z'_N(i,j)$ is the configuration after the collision between the particles $i$ and $j$. By using the same procedure as before we can derive a BBGKY hierarchy for the $j$-particle distributions $f_j^N$. By taking the limit $N \to \infty, \alpha \to 0, N\alpha \to l$, (note that here $\varepsilon$ is assumed fixed), we obtain the Boltzmann hierarchy for the family $f_j$ the $j$-particle distribution. In the hypothesis of propagation of chaos (i.e. the factorization of $f_j$) we obtain the Boltzmann equation (1) introduced in Sect. 1.

## 3 The homogeneous case

In the following, we suppose, for sake of simplicity, that for $t = 0$ the velocity support of $f$ is bounded, i.e.

$$f(v,0) = 0 \qquad \text{if} \qquad |v| > v_0.$$

As follows by the collision rule, this property is preserved by the dynamics.

In the homogeneous case, Eq. (1) becomes:

$$\partial_t f(v,t) = l\int dv_1|v - v_1|\left(\frac{f(v^*,t)f(v_1^*,t)}{(1-2\varepsilon)^2} - f(v,t)f(v_1,t)\right). \tag{11}$$

By neglecting the loss term:

$$\partial_t f(v,t) = l\int dv_1|v - v_1|\left(\tfrac{f(v^*,t)f(v_1^*,t)}{(1-2\varepsilon)^2}\right)$$

$$= \frac{l}{1-\varepsilon}\int dv_1^*\ |v^* - v_1^*)|\ f(v^*,t)f(v_1^*,t) \tag{12}$$

Then

$$\frac{d}{dt}\left|\left|f(\cdot,t)\right|\right|_\infty \le 4lv_0\left|\left|f(\cdot,t)\right|\right|_\infty \tag{13}$$

and

$$\left|\left|f(\cdot,t)\right|\right|_\infty \le \left|\left|f\ (\cdot,0)\right|\right|_\infty e^{4lv_0t} \tag{14}$$

By the global estimate (13) it is possible to obtain an existence and uniqueness result for Eq. (11) in $L_\infty$ by standard methods.

Now we consider the limit $l \to \infty, \varepsilon \to 0, l\varepsilon = \lambda$, and we prove that the solutions of (11) converge weakly, in the sense of the weak convergence of the measures, to the homogeneous solutions of Eq. (3) which verify:

$$\partial_t f(v,t) = -\lambda\partial_v(fF)(v,t) \tag{15}$$

where:

$$F(v,t) = \int d\overline{v}(\overline{v} - v)|\overline{v} - v|f(\overline{v},t). \tag{16}$$

Eq. (15-16) has been studied in [4], where, in particular, it is proved an existence and uniqueness of the solutions in the space of positive and bounded measures.

Let $\phi \in \mathbf{C}_0^\infty(\mathbb{R})$ a test function, $\lambda = l\varepsilon$ and $f_\varepsilon$ the solution of Eq. (11):
$\int \phi(v)f_\varepsilon(v,t)dv = \int \phi(v)f(v,0)+$

$$+\lambda\int_0^t ds\int dvdv_1|v - v_1|f_\varepsilon(v,S)f_\varepsilon(v_1,S)\frac{\phi(v') - \phi(v)}{\varepsilon}, \tag{17}$$

where $v' = v - \varepsilon(v - v_1)$. Then

$$\begin{aligned} &\int \phi(v)f_\varepsilon(v,t)dv = \int \phi(v)f(v,0)+ \\ &-\lambda\int_0^t ds\int dvdv_1(v - v_1)|v - v_1|f_\varepsilon(v,s)f_\varepsilon(v_1,s)\partial_v\phi(v)+ \\ &+O(\varepsilon v_0^3\|\partial_v^2\phi\|_\infty) \end{aligned} \tag{18}$$

At this point one can use standard compactness argument to prove:

**Theorem 3.1** *Let $f(v,0) \in \mathbf{L}_\infty(\mathbb{R})$ with compact support and $f_\varepsilon(v,t)$ be the solution of (11) with initial condition $f(v,0)$, where $l\frac{\lambda}{\varepsilon}$. Then $f_\varepsilon(v,t)$ converges weakly, as $\varepsilon \to 0$, to the unique solution of (15-16) with the same initial condition.*

## 4 The Cauchy problem for the inhomogeneous Boltzmann Equation

We shall now prove a global existence and uniqueness result for the solution of Eq. (1) for small values of $l$ compared with the $\mathbf{L}_1$ norm of the initial datum.

**Theorem 4.1** *Let $f_0 \in \mathbf{L}_\infty(\mathbb{R}^2), f_0 \geq 0, \|f_0\|_1 = 1, f_0(x,v) = 0$ if $|v| \geq v_0$. Then, for $l < 1/8$, there exist an unique mild, bounded solution in $\mathbf{L}_\infty$ of (1) with initial datum given by $f_0$.*

**Proof.** Let $f^\#(x,v,t) = f(x+vt,v,t)$. It is easy to realize that $f^\#$ satisfies

$$\frac{d}{dt} f^\#(x,v,t) = Q^\#(f,f), \tag{19}$$

where $Q^\#(f,f) = Q(f,f)(x+vt,v)$ according to the previous notation and

$$Q(f,f) = l \int dv_1 |v - v_1| \left( \frac{f(x,v^*,t) f(x,v_1^*,t)}{1-2\varepsilon} - f(x,v,t) f(x,v_1,t) \right) \tag{20}$$

By integrating in the time:

$$f^\#(x,v,t) = f(x,v,0) + l \int_0^t ds Q(f,f)(x,v,s). \tag{21}$$

Let

$$F(x,v) = \sup_{t \geq 0} f^\#(x,v,t). \tag{22}$$

From (21):

$$\begin{aligned} f^\#(x,v,t) &\leq f(x,v,0) + \\ & l \int_0^t ds \int dv_1 \frac{|v - v_1|}{(1-2\varepsilon)^2} f(x+vs,v^*,s) f(x,vs,v_1^*,s). \end{aligned} \tag{23}$$

By considering that:

$$f(x+vs,v^*,s) = f^\#(x+(v-v^*)s, \leq F(x+(v-v^*)s,v^*)$$

$$f(x+vs,v_1^*,s) = f^\#(x+(v-v_1^*)s, \leq F(x+(v-v_1^*)s,v_1^*) \tag{24}$$

From (23) we have:

$$F(x,v) \le f(x,v,0) + \\ l\int_0^\infty ds \int dv_1 \frac{|v-v_1|}{(1-2\varepsilon)^2} F(x+(v-v^*)s, v^*) F(x+(v-v_1^*)s, v_1^*), \tag{25}$$

and

$$\begin{aligned} \int dx dv F(x,v) &\le 1 + l\int_0^\infty ds \int dx dv dv_1 \frac{|v-v_1|}{(1-2\varepsilon)^2} \\ &= 1 + l\int dz dy \frac{dv dv_1}{1-2\varepsilon} F(y,v^*) F(z,v_1^*) \end{aligned} \tag{26}$$

Putting $y = x + (v - v^*)s$ and $z = y + (v^* - v_1^*)s$, by elementary computations we obtain:

$$\begin{aligned} \int dx dv F(x,v) &\le 1 + l\int dz dy \frac{dv dv_1}{(1-2\varepsilon)} \frac{|v-v_1|,}{|v^*-v_1^*|} F(y,v_1^*) F(z,v^*) = \\ &= 1 + l\int dz dy \frac{dv dv_1}{1-2\varepsilon} F(y,v^*) F(z,v_1^*). \end{aligned} \tag{27}$$

Then

$$\|F\|_1 \le 1 + l\int dz dy dv^* dv_1^* F(y,v_1^*) F(z,v^*) = 1 + l\|F\|_1^2, \tag{28}$$

and therefore $F \le 2$ if $l < \frac{1}{4}$.

Now we are able to prove an $\mathbf{L}_\infty$ a priori estimate on $f(x,v,t)$.

Let

$$G = \sup_{x,v,t\ge 0} f(x,v,t).$$

From (23):

$$G \le \|f_0\|_\infty + lG \sup_{x,v} \int_0^\infty ds \int dv_1 \frac{|v-v_1|}{(1-2\varepsilon)^2} FG \le \|f_0\|_\infty + 8lG. \tag{29}$$

then, with the change of variable $y = x + (v - v_1^*)s$, by integrating in $dy dv_1^* = |v - v_1| \left(\frac{1-\varepsilon}{1-2\varepsilon}\right)^2 ds\, dv_1$:

$$G \le \|f_0\|_\infty + \frac{l}{(1-\varepsilon)^2} FG \le \|f_0\|_\infty + 8lG. \tag{30}$$

Therefore

$$G \le \frac{\|f_0\|_\infty}{1-8l}.$$

With this a priori bound the construction of the solutions is standard. □

Theorem 4.1 follows a strategy proposed by L. Arkeryd in [1] for one dimensional non dissipative systems. The main difficulty in extending this result to arbitrary $l$ is the lack of an entropy control, which, in the non dissipative case, allows us to extend this kind of results to arbitrary $l$ by using a classical argument. Unfortunately in the present context we do not have an H-Theorem.

It is not worthless to mention that also Bony's approach (see [6]) to one-dimensional kinetic models which do not make use the entropy functional, does not prevent, in our case, a blow up in a finite time. However we can exclude a total concentration of the solution in a finite time. Namely suppose that there exists a critical time $t^*$ such that:

$$f(v,v,t) \to \delta(v)\delta(x-x_0) \tag{31}$$

where we are assuming that $\int f_0(x,v)dxdv = 1$ and the above convergence is understood in the sense of the weak convergence of the probability measures, We denote by $Q^+(f,f)$ and by $fQ^-(f)$ the gain and loss term respectively in Eq. (1). By Eq. (19) we have:

$$Q^+(f,f) = \int dv_1 |v - v_1| \frac{f(x,v^*,t)f(x,v_1^*,t)}{(1-2\varepsilon)^2} \tag{32}$$

$$Q^-(f) = \int dv_1 |v - v_1| f(x,v_1,t). \tag{33}$$

from which

$$f(x,v,t) \geq \exp(-\int_0^t ds \int dv_1 |v-v_1| f(x-v(t-s),v_1,s) f_0(x,v) \tag{34}$$

Denoting by $dm_0 = f_0 dxdv$, taking any measurable set $A$ and using the Jensen inequality:

$$\begin{aligned} &\textstyle\int_A f(x,v,t)dxdv \geq m_0(A)\cdot \\ &\cdot \exp(\tfrac{1}{m_0(A)} \textstyle\int_0^t ds \int dx \int dv \int dv_1 |v-v_1| f(x-v(t-s),v_1,s) f_0(x,v) \\ &\geq \exp(\tfrac{-ct}{m_0(A)} \|f_0\|_\infty) \end{aligned} \tag{35}$$

The above inequality can be used to exclude the occurrence of a total concentration (31). Indeed first notice that a mild solution can be constructed under the same hypotheses of Theorem 4.1 for a large $l$ but for a short time. Suppose now that there exists a critical time $t^*$ for which (31) holds. Then choose $A$ an open set not containing $x_0$ and such that $m_0(A) > 0$. By condition (31) we have:

$$\int_A f(x, v, t) \to 0 \tag{36}$$

as $t \to t^*$ which contradicts (35).

Unfortunately we are not able to show the absence of the occurrence of other singularities but we believe that the solutions of Eq. (1) have a much more regular behavior than those of Eq. (2-3).

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