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CIRCLE PACKINGS OF MAPS – THE EUCLIDEAN CASE†

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ABSTRACT. In an earlier work, the author extended the Andreev-Koebe-Thurston circle packing theorem. Additionally, a polynomial time algorithm for constructing primal-dual circle packings of arbitrary (essentially) 3-connected maps was found. In this note, additional details concerning surfaces of constant curvature 0 (with special emphasis on planar graphs where a slightly different treatment is necessary) are presented.

1 Introduction

Let Σ be a surface. A *map* on Σ is a pair (G, Σ) where G is a connected graph that is 2-cell embedded in Σ. Given a map $M = (G, \Sigma)$, a *circle packing* of M is a set of closed balls (called *circles*) C_v, $v \in V(G)$, in a Riemannian surface Σ' of constant curvature +1, 0, or -1 that is homeomorphic to Σ such that the following conditions are fulfilled:

(i) Each circle C_v is a ball of radius r_v with respect to the geodesic distance in Σ', and the interiors of these circles are pairwise disjoint open disks.

(ii) For each edge $uv \in E(G)$, the circles C_u and C_v touch.

(iii) By putting a vertex v° in the center of each circle C_v and joining v° by geodesics with all points on the boundary of C_v where the other

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circles touch C_v (or where C_v touches itself), we get a map on Σ' which is isomorphic to M.

Because of (iii) we also say to have a *circle packing representation* of M. Simultaneous circle packing representations of a map M and its dual map M^* are called a *primal-dual circle packing representation* of M if

(iv) For any two edges $e = uv \in E(M)$ and $e^* = u^*v^* \in E(M^*)$ which are dual to each other, the circles C_u, C_v corresponding to e touch at the same point as the circles C_{u^*}, C_{v^*} of e^*, and C_u, C_{u^*} cross each other at that point at the right angle.

Having a primal-dual circle packing representation, each pair of dual edges intersects at the right angle.

It was proved by Koebe [7], Andreev [1, 2], and Thurston [13] that if M is a triangulation of the sphere, then it admits a circle packing representation. The proofs of Andreev and Thurston are existential (using a fixed point theorem) but Colin de Verdière [4, 5] found a constructive proof by means of a convergent process (also for more general surfaces). Pulleyblank and Rote (private communication) and Brightwell and Scheinerman [3] proved existence of primal-dual circle packings of 3-connected planar graphs. The author [10] extended these results by characterizing maps on general surfaces that admit primal-dual circle packing representation. In particular, every map (on an arbitrary closed surface) with a 3-connected graph has a primal-dual circle packing representation.

The presentation in [10] focused on the hyperbolic case, the constant negative curvature. The proofs for the Euclidean case (curvature 0) were omitted, and in this note related to the Colloquium talk of the author, we give more detailed presentation of the Euclidean case. In fact, we concentrate on the case of primal-dual circle packing representation of planar graphs in the Euclidean plane since that case needs slightly different treatment than circle packings on closed surfaces. We present a polynomial time algorithm which gets as the input an essentially 3-connected map M on a flat surface (the torus, the Klein bottle, or the plane) and a rational number $\varepsilon > 0$, and finds an ε-approximation for a circle packing of M into a surface of constant curvature 0. The time used by this algorithm is polynomial in the size of the input (which is defined as $|E(M)| + \max\{1, \lceil\log(1/\varepsilon)\rceil\}$).

The proofs establishing existence and uniqueness of primal-dual circle packings are elementary. The basic idea relies on the interpretation (due to Lovász) of Thurston's proof, a version of which was also used by Brightwell and Scheinerman [3]. It is the details in the algorithm that require more work in order to show that the worst case running time is polynomial. A new result in this note is the extension of the primal-dual circle packing theorem to a more general class of plane graphs if we do not insist to have the circle corresponding to the unbounded face; see Theorem 3.9.

There are many applications of circle packing representations in computational geometry, graph drawing, computer graphics (cf., e.g., [6, 8, 9]), as well as in complex analysis (cf., e.g., [12]).

2 Primal-dual circle packings

Let Σ be a surface of constant curvature 0 which is isomorphic to the torus, the Klein bottle, or the Euclidean plane. Let $M_0 = (G_0, \Sigma)$ be a map on Σ. Define a new map $M = (G, \Sigma)$ whose vertices are the vertices of G_0 together with the faces of M_0, and whose edges correspond to the vertex-face incidence in M_0. The embedding of G is obtained simply by putting a vertex in each face F of M_0 and joining it to all the vertices on the boundary of F. If a vertex of G_0 appears more than once on the boundary of the face, then we get multiple edges at F but their order around F is determined by the order of the vertices on the boundary of F. The map M and the graph G are called the *vertex-face map* and the *vertex-face graph*, respectively. (Sometimes also the name *angle map* and *angle graph* is used.) Note that G is bipartite and that every face of M is bounded by precisely four edges of G.

From now on we assume that M_0 is a given map on Σ and that M and G are its vertex-face map and vertex-face graph, respectively. We will use the notation $V = V(G)$, and will denote by n and m the number of vertices and edges of G, respectively. If Σ is the torus or the Klein bottle, then by Euler's formula

$$m = 2n. \tag{1}$$

Similarly, if Σ is the plane, then

$$m = 2n - 4. \tag{2}$$

If $S, T \subseteq V(G)$, then $E(S)$ denotes the set of edges with both endpoints in S, and $E(S,T)$ is the set of edges with one endpoint in S and the other in T. Although $E(S,T) = E(T,S)$, we emphasize that, in order to simplify the notation, $uv \in E(S,T)$ will not only mean the membership but will also implicitly assume that $u \in S$, $v \in T$.

If Σ is the plane, then one of the vertices of the vertex-face graph corresponds to the unbounded face, and we refer to it as the *vertex at infinity.* It is convenient to consider circle packings in the *extended plane* (the plane together with a point ∞ which we call *infinity*). Then we allow that one of the circles of a circle packing, denoted by C_ω, behaves differently. Instead of (i), we require that none of the circles intersects the exterior of C_ω. We call C_ω a circle *centered at infinity*. To get the corresponding circle packing representation in (iii), each edge from a vertex v to the vertex of C_ω is represented by the half-line from the center of C_v through $C_v \cap C_\omega$ (towards

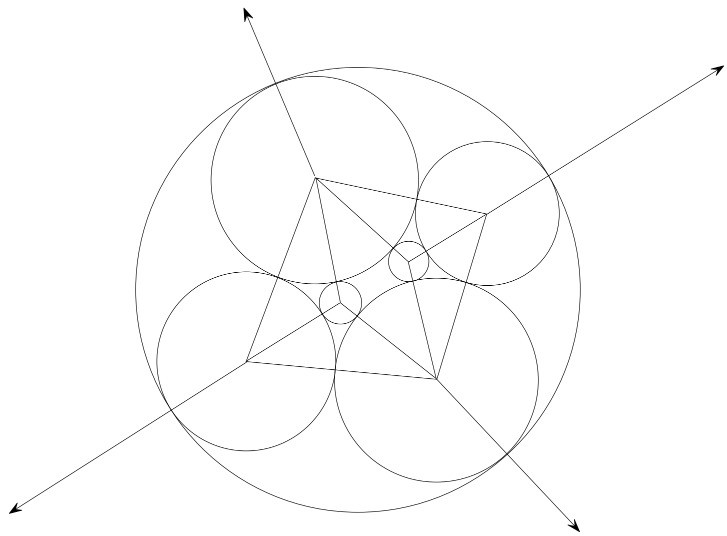

Figure 1: *A CP with a circle centered at infinity*

infinity). See Figure 1 for an example of a CP representation with a circle centered at infinity.

Similarly we extend the notion of primal-dual circle packings in the plane to allow the circle at infinity, and we assume that the vertex at infinity is the vertex of G corresponding to the unbounded face of G_0. Lemma 2.1 below shows that the last assumption may as well be omitted.

Let us view $\boldsymbol{R}^2$ as the complex plane $\boldsymbol{C}$, and the extended plane as $\boldsymbol{C}^* = \boldsymbol{C} \cup \{\infty\}$. Consider transformations $w : \boldsymbol{C}^* \to \boldsymbol{C}^*$ of the following form:

$$w(z) = \frac{az + b}{cz + d}, \qquad ad - bc \neq 0,$$

where $w(\infty) = a/c$ if $c \neq 0$ and $w(\infty) = \infty$ if $c = 0$. Also, $w(-d/c) = \infty$. These maps are called *fractional linear transformations* or *Möbius transformations*. They map circles and lines to circles and lines in $\boldsymbol{C}^*$ (lines in $\boldsymbol{C}^*$ correspond to usual lines in the plane together with the point ∞). In particular, they map (primal-dual) circle packings to (primal-dual) circle packings. Now, it is easy to see the following:

Lemma 2.1 *If a graph G has a circle packing in the plane and v is a vertex of G, then there is a circle packing representation of G such that the circle corresponding to v is centered at infinity.*

Let us now return to the general case. Having a primal-dual circle packing representation of M_0 in Σ, we have a circle for each vertex v of G. Let

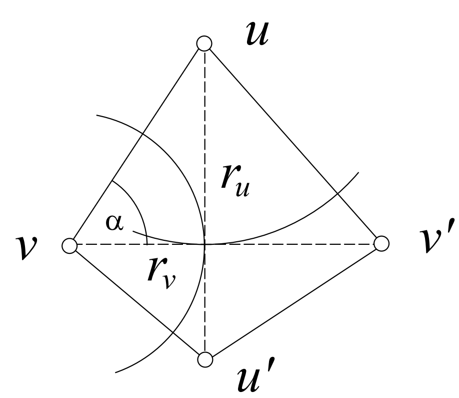

Figure 2: *A basic quadrangle*

r_v be the radius of that circle. Clearly, the primal-dual circle packing representation in Σ gives rise to a straight-line representation of M. Consider a vertex v of M. It is surrounded by quadrilaterals (called *basic quadrangles*). If $vuv'u'$ is one of them (cf. Figure 2), then its diagonals are perpendicular and have length $r_v + r_{v'}$ and $r_u + r_{u'}$, respectively. The angle α shown in Figure 2 is equal to:

$$\alpha = \mathrm{arctg}(r_u/r_v). \tag{3}$$

Since the total sum of the angles around a vertex is 2π, we have a necessary condition for a set of radii $r = (r_v \mid v \in V(G))$ to be the radii of a primal-dual circle packing:

$$\sum_{vu \in E(G)} \mathrm{arctg}(r_u/r_v) = \pi\,, \quad v \in V(G) \tag{4}$$

where the sum is taken over all edges vu that are incident to v in G. It is important that (4) is also sufficient, as shown by Brightwell and Scheinerman [3] in the planar case and by Mohar [10] in the closed surface case.

Proposition 2.2 *Let M be the vertex-face map of a map M_0 on a surface Σ of constant curvature 0. Let G be the vertex-face graph of M. Then $r = (r_v \mid v \in V(G))$ are the radii of a primal-dual circle packing representation of M if and only if $r_v > 0$, $v \in V(G)$, and the angle condition* (4) *is satisfied.*

3 Planar graphs

Let G_0 be a given 2-connected plane graph. In this section we let G be the graph which is obtained from the vertex-face graph of G_0 by deleting the vertex corresponding to the unbounded face of G_0. Suppose that the unbounded face of G_0 contains k vertices $v_1, \dots, v_k$ on its boundary. Denote by n, m, f the number of vertices, edges, and faces of G, respectively. Then (2) implies

$$m = 2n - k - 2 \quad \text{and} \quad f = n - k\,.$$

(Observe that (2) refers to the vertex face graph $G + \omega$ which has $n + 1$ vertices and $m + k$ edges in the current notation.) Note that G is a bipartite plane graph and that G is simple (since G_0 is 2-connected), although we do not require G_0 be a simple graph.

Given a function $r = (r_v \mid v \in V)$, where each $r_v > 0$, we define

$$\varphi_v = \sum_{u \sim v} \operatorname{arctg} \frac{r_u}{r_v}$$

where the sum is taken over all vertices u that are adjacent to v in G. Clearly, φ_v is equal to one half of the total angle of basic quadrangles around v. To measure the difference from the expected value π (or $\frac{k-2}{2k}\pi$ if $v = v_i$), we introduce

$$\vartheta_v = \begin{cases} \varphi_v - \pi, & \text{if } v \notin \{v_1, \dots, v_k\} \\ \varphi_v - \frac{k-2}{2k}\pi, & \text{if } v \in \{v_1, \dots, v_k\} \end{cases} . \tag{5}$$

Denote by $\Theta(r) = (\vartheta_v;\, v \in V)$.

Lemma 3.1 $\sum_{v \in V} \vartheta_v = 0$.

Proof. We will use here and in later proofs the well known identity

$$\operatorname{arctg}(x) + \operatorname{arctg}(\frac{1}{x}) = \frac{\pi}{2}\,.$$

It follows easily:

$$\begin{aligned} \sum_{v \in V} \vartheta_v &= \sum_{v \in V} \varphi_v - \pi(n-k) - k\,\frac{k-2}{2k}\pi \\ &= \frac{\pi}{2}e - (2n - 2k + k - 2)\frac{\pi}{2} = 0\,. \end{aligned}$$

□

The following lemma is obvious.

Lemma 3.2 *The functions $\vartheta_v = \vartheta_v(r)$ are continuous and differentiable. Moreover, $\partial\vartheta_v/\partial r_v < 0$, $\partial\vartheta_v/\partial r_u > 0$, if u is adjacent to v in G, and $\partial\vartheta_v/\partial r_u = 0$, otherwise.*

Lemma 3.3 *Let $S \subseteq V$, $S \neq \emptyset$, $S \neq \{v_i\}$ $(1 \leq i \leq k)$, and let $t = |S \cap \{v_1 \ldots, v_k\}|$. If $E(S)$ denotes the set of edges of G with both ends in S, then*

$$2|S| - |E(S)| \geq t + 2\,. \tag{6}$$

If $t = k$ and either the graph $G_S = (S, E(S))$ induced on S is disconnected, or at least one bounded face of G_S is not a quadrilateral, or the unbounded face of G_S is not of size $2k$, then

$$2|S| - |E(S)| \geq t + 3\,. \tag{7}$$

Proof. Let us first assume that G_S is connected. Then we obviously have the following. If $t = 0$ and $|S| = 1$ then

$$2|S| - |E(S)| = 2 = t + 2\,. \tag{8}$$

If $t = 1$ and $|S| = 1$ then

$$2|S| - |E(S)| = 2\,. \tag{9}$$

If $|S| = 2$, then $t \leq 1$. Hence

$$2|S| - |E(S)| \geq 3 \geq t + 2\,. \tag{10}$$

If $|S| \geq 3$ then consider G_S as the plane graph. Since G_S is bipartite and simple, all its faces are of size 4 or more. Moreover, the unbounded face contains the t vertices of $S \cap \{v_1, \ldots, v_k\}$, and is therefore of size at least $2t$. (We do not insist that the unbounded face is simple!) Consequently, counting the number of edges on the boundaries of faces of G_S yields:

$$4(|F(G_S)| - 1) + 2t \leq 2|E(S)| \tag{11}$$

where $F(G_S)$ denotes the set of faces of G_S. By Euler's formula for G_S and (11), we get (6). In the case when G_S is connected, $t = k$, and at least one of the bounded faces of G_S is not a quadrilateral, or the unbounded face is of size greater than $2k$, (11) can easily be improved to (7).

It follows by (8)–(10) that in the case when G_S is not connected the inequality (7) holds. Clearly, this implies (6). □

Lemma 3.4 *Let $r = (r_v;\, v \in V)$, $S \subset V$, $S \neq \emptyset$, and $\alpha > 0$ be given. Define r' by $r'_v = \alpha r_v$ if $v \in S$, and $r'_v = r_v$ otherwise. Let $(\vartheta'_v;\, v \in V) = \Theta(r')$, and let $f(\alpha) = \sum_{v \in S}(\vartheta_v - \vartheta'_v)$. Then*

(a) *If $\alpha \geq 1$ then $\vartheta'_v \leq \vartheta_v$ if $v \in S$, and $\vartheta'_v \geq \vartheta_v$ if $v \notin S$.*

(b)

$$f(\alpha) = \sum_{v \in S} \sum_{u \notin S, vu \in E} (\operatorname{arctg} \frac{r_u}{r_v} - \operatorname{arctg} \frac{r_u}{\alpha r_v})\,.$$

In particular, $f(\alpha)$ is monotone increasing.

(c) *If* $M = \lim_{\alpha\to\infty} f(\alpha)$ *then*

$$\sum_{v\in S} \vartheta_v \le M < \frac{\pi}{2}|E(S, V\backslash S)|.$$

(d) *If* $\alpha \ge 2 + \max_{u\notin S} r_u / \min_{v\in S} r_v$, *then*

$$f(\alpha) \ge \frac{M}{2}\ .$$

Proof. The assertion (a) is obvious. By (a) it is clear that the limit M is well defined and that it is equal to

$$M = \sum_{vu\in E(S,V\backslash S)} \operatorname{arctg}\frac{r_u}{r_v} \le \frac{\pi}{2}|E(S,V\backslash S)|\,. \tag{12}$$

To get the lower bound on M, we use the fact that $\vartheta_v = \varphi_v - \pi$, or $\vartheta_v = \varphi_v - \frac{k-2}{2k}\pi > \varphi_v - \pi$, but the latter happens only for the $t \le k$ vertices of $S \cap \{v_1, \dots, v_k\}$. We will write $s = |S|$ and $e_S = |E(S)|$. Then

$$\begin{aligned}
\sum_{v\in S}\vartheta_v &= \sum_{v\in S}\varphi_v - \pi(s-t) - t\,\frac{k-2}{2k}\pi \\
&= \sum_{v\in S}\sum_{vu\in E}\operatorname{arctg}\frac{r_u}{r_v} - \Big(s - t + t\,\frac{k-2}{2k}\Big)\pi \\
&= \frac{\pi}{2}e_S + M - \Big(s - t + t\,\frac{k-2}{2k}\Big)\pi \\
&= M - (2s - e_S - t - \frac{2t}{k})\pi\,. && (13)
\end{aligned}$$

By applying (6) in the above inequality we get the required bound. There is the case when $t = 1$ and $s = 1$ which is not covered by (6). But in this case $2s - e_S - t - 2t/k = 1 - 2/k \ge 0$ which yields the same conclusion.

To prove (c) we use (a) and (12). It suffices to see that

$$\operatorname{arctg}\frac{r_u}{r_v} - \operatorname{arctg}\frac{r_u}{\alpha r_v} \ge \frac{1}{2}\operatorname{arctg}\frac{r_u}{r_v} \tag{14}$$

for every $v \in S, u \notin S$. Let $x = r_u/r_v$ and $y = x/\alpha$. Then (14) is equivalent to $\operatorname{arctg} x \ge 2\operatorname{arctg} y$. Since $0 < y < 1$, we have $2\operatorname{arctg} y = \operatorname{arctg}\frac{2y}{1-y^2}$, and the previous inequality reduces to $x \ge 2y/(1-y^2)$. This is equivalent to $\alpha \ge \sqrt{x^2+1} + 1$. But this is true since $\alpha \ge x + 2$ by assumption. □

Given r, define

$$\mu(r) = \sum_{v\in V}\vartheta_v^2\,.$$

Order the vertices of G such that $\vartheta_{u_1} \geq \vartheta_{u_2} \geq \cdots \geq \vartheta_{u_n}$. Let

$$\sigma(r) = \max_{1 \leq i < n} (\vartheta_{u_i} - \vartheta_{u_{i+1}}), \tag{15}$$

and let t be the smallest index i where the maximum in (15) is attained. Set $S = S(r) = \{u_1, \ldots, u_t\}$, and let r' be defined by $r'_v = \alpha r_v$ if $v \in S$, and $r'_v = r_v$ otherwise. Let $(\vartheta'_v; v \in V) = \Theta(r')$, and let $f(\alpha) = \sum_{v \in S}(\vartheta_v - \vartheta'_v)$. Call α *suitable* if

(a) $\vartheta'_v \geq \vartheta'_u$ for all $v \in S$ and $u \notin S$, and

(b) $f(\alpha) \geq \frac{1}{3} \min \{\sigma(r), \sum_{v \in S} \vartheta_v\}$.

Lemma 3.4(c) shows that a suitable α always exists.

Next we describe an algorithm for the following problem:

Instance: A 2-connected plane graph G_0, $\varepsilon > 0$.

Task: Find positive numbers $r = (r_v; v \in V)$ for the corresponding graph G such that $\mu(r) \leq \varepsilon$.

ALGORITHM A:

1. Construct the vertex-face graph and remove its vertex at infinity to form the graph G.
2. Set $r_v = 1$, $v \in V$.
3. **while** $\mu(r) > \varepsilon$ **do**
   - 3.1 Determine $\sigma = \sigma(r)$ and the set $S \subset V$.
   - 3.2 Find a suitable α by bisection. Since all the computations are only approximative (p binary digits), we perform the bisection as follows:
     $\alpha_0 := 1,\ \alpha_1 := 2(\max_{u \notin S} r_u + 2)$
     Comment: Note that $\min_{v \in V} r_v = 1$.
     repeat
       $\alpha := \frac{\alpha_0 + \alpha_1}{2}$
       Compute $r',\ \Theta(r') = (\vartheta'_v \mid v \in V)$.
       if $\vartheta'_v < \vartheta'_u + \frac{1}{7}\sigma$ for some $v \in S, u \notin S$
       then $\alpha_1 := \alpha$ **else** $\alpha_0 := \alpha$
     until α is suitable.
   - 3.3 $\rho := \min_{v \in V} r'_v$ and $r_v := r'_v/\rho$, $v \in V$.
4. Output r and $\Theta(r)$.

This algorithm needs some comments:

- All the arithmetic in Algorithm A should be performed with "large" precision. Lemma 3.8 below can be used to show that $p = 10n\log_2 n + \lceil \log(1/\varepsilon) \rceil$ significant binary digits will suffice for that purpose. However, for most practical issues and applications, the built-in computer arithmetic should suffice.

- Instead of the bisection in Step 3 of Algorithm A, one can use Newton's iteration. It is clear that this change would improve the performance. We have not used it in Step 3 since the formal proof needs an additional argument in that case.

- The computation of ϑ_v and ϑ'_v (to the required precision p) is also polynomial since the Taylor series of $\mathrm{arctg}(x)$ converge fast enough.

The following two lemmas show that the number of repetitions of Step 3 in Algorithm A is bounded by a polynomial in n and $\lceil \log(1/\varepsilon) \rceil$.

Lemma 3.5 *If $r_v = 1$ for each $v \in V$ and G has no vertices of degree 0 or 1, then*

$$\mu(r) < \pi^2 n^2.$$

Proof. In this case we have $\varphi_v = \deg(v)\frac{\pi}{4} \geq \frac{\pi}{2}$. It follows that $|\vartheta_v| \leq \frac{\pi}{4}\deg(v)$. Finally:

$$\mu(r) \leq \frac{\pi^2}{16}\sum_{v\in V}\deg(v)^2 < \frac{\pi^2}{16}\left(\sum_{v\in V}\deg(v)\right)^2 = \frac{\pi^2}{4}|E(G)|^2 .$$

Now (2) completes the proof. □

Lemma 3.6 *If r' is the new value for the function r obtained by the algorithm, then*

$$\mu(r') \leq \left(1 - \frac{1}{6n^3}\right)\mu(r) .$$

Proof. Using the notation of the Algorithm, let $t_1 = \min_{v\in S}\vartheta_v$, $t_2 = \max_{v\notin S}\vartheta_v$. Then $t_1 - t_2 \geq \sigma$. Since α is suitable, there is a number t_3 between t_2 and t_1, such that for every $v \in S, u \notin S$, $\vartheta'_v \geq t_3 \geq \vartheta'_u$. Since $\alpha > 1$ we have $\vartheta_v \geq \vartheta'_v$ for $v \in S$, and $\vartheta_u \leq \vartheta'_u$ for $u \notin S$. Finally,

$$\begin{aligned}
\mu(r) - \mu(r') &= \sum_{v\in V}(\vartheta_v^2 - \vartheta_v'^2) \\
&= \sum_{v\in S}(\vartheta_v + \vartheta'_v)(\vartheta_v - \vartheta'_v) + \sum_{u\notin S}(\vartheta_u + \vartheta'_u)(\vartheta_u - \vartheta'_u) \\
&\geq \sum_{v\in S}(t_1 + t_3)(\vartheta_v - \vartheta'_v) + \sum_{u\notin S}(t_2 + t_3)(\vartheta_u - \vartheta'_u)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{v\in S}(t_1-t_2)(\vartheta_v-\vartheta'_v) \qquad \text{(by Lemma 3.1)}\\
&\geq \sigma\sum_{v\in S}(\vartheta_v-\vartheta'_v)=\sigma f(\alpha)\\
&\geq \frac{\sigma}{3}\min\Big\{\sigma,\sum_{v\in S}\vartheta_v\Big\}\geq\frac{\sigma^2}{6}\,.
\end{aligned}$$

In the last inequality we used the fact that $\sum_{v\in S}\vartheta_v \geq \sigma/2$ which is left to be verified by the reader as an exercise.

To prove the inequality of the lemma we combine the above bound with the following one:

$$\begin{aligned}
\sigma &\geq \frac{1}{n-1}\Big(\max_{v\in V}\vartheta_v-\min_{v\in V}\vartheta_v\Big)\\
&> \frac{1}{n}\sqrt{\vartheta_{u_1}^2+\vartheta_{u_n}^2}\geq\frac{1}{n}\sqrt{\frac{1}{n}\sum_{v\in V}\vartheta_v^2}=n^{-3/2}\sqrt{\mu(r)}\,.
\end{aligned}$$

□

Suppose that G satisfies the following property. If S is a proper subset of vertices of G, which contains all vertices $\{v_1,\ldots,v_k\}$, then the graph G_S induced on S is either disconnected, or contains a bounded face which is not a quadrilateral, or the unbounded face contains more than $2k$ edges on its boundary (counted twice if an edge appears twice on its boundary). Then G is said to be *almost 3-connected*. It is easy to see that if G_0 is 3-connected then G is almost 3-connected. But there are other examples of graphs G_0 which give rise to almost 3-connected G. For example, 2-connected outerplanar graphs. Their combinatorial characterization is as follows. Let $\tilde{G}$ be a 3-connected plane graph, and let C be a cycle of $\tilde{G}$. If G_0 is the plane subgraph of $\tilde{G}$ which consists of C and and all edges in the interior of the disk bounded by C, then we say that G_0 is a *cycle graph*.

Proposition 3.7 *Let G_0 be a 2-connected plane graph, and let G be the subgraph of the vertex-face graph of G_0 obtained by removing the vertex at the infinity. Then the following are equivalent:*

(a) *G_0 is a cycle graph.*

(b) *The graph $\tilde{G}$ obtained from G_0 by adding a new vertex joined to all vertices on the outer face of G_0 is 3-connected.*

(c) *G is almost 3-connected.*

Proof. Equivalence of (a) and (b) is easy to see and is left to the reader. Let w be the vertex that was added to G_0, and let G' be the vertex-face graph of

$\tilde{G}$. If $S \subset V(G)$, we denote by S' the set of vertices of G' consisting of S, w, and all faces of $\tilde{G}$ containing w. If G is not almost 3-connected, then there is a set $S \subset V(G)$ such that $v_1, \ldots, v_k \in S$, and the corresponding subgraph $G'_{S'}$ of G' is a quadrangulation. It was proved in [3] that this implies that $\tilde{G}$ is not 3-connected. This shows that (b) implies (c).

To show the converse, suppose that vertices x, y form a vertex 2-separation of $\tilde{G}$. Then, clearly, $w \notin \{x, y\}$. There are distinct faces ν, τ of $\tilde{G}$ which contain edges in distinct components of $\tilde{G} - x - y$. Then $x\nu y\tau$ is a 4-cycle of G such that the set T of vertices of G inside this 4-cycle is nonempty. Let $S = V(G) \backslash T$. This set shows that G is not almost 3-connected. This proves that (c) implies (b). □

For our purpose the following lemma is important.

Lemma 3.8 *Let G be an almost 3-connected graph. Suppose that $\mu(r) \leq \varepsilon \leq \frac{\pi^2}{4n^3}$ and that $\min_{v \in V} r_v = 1$. Then*

$$\max_{v \in V} r_v \leq (2n^2)^{n-1}.$$

Proof. It suffices to show that for an arbitrary nonempty proper subset $S \subset V$ of vertices we have

$$\min_{v \in S} r_v \leq 2n^2 \max_{u \notin S} r_u. \tag{16}$$

Assume that (16) does not hold for S. Let $a = \min_{v \in S} r_v$ and let $b = \max_{u \notin S} r_u$. Also denote by $s = |S|$ and $t = |S \cap \{v_1, \ldots, v_k\}|$. By (13) we have:

$$\begin{aligned} \sum_{v \in S} \vartheta_v &= \sum_{vu \in E(S, V \backslash S)} \operatorname{arctg} \frac{r_u}{r_v} - \Big(2s - |E(S)| - t - \frac{2t}{k}\Big)\pi \\ &\leq |E(S, V \backslash S)| \, \frac{b}{a} - \tau\pi \leq 2n \, \frac{b}{a} - \tau\pi \end{aligned}$$

where $\tau = 2s - |E(S)| - t - \frac{2t}{k}$. If $t \leq 1$ and $s \leq 2$, then we have

$$\tau \geq \frac{1}{3} \tag{17}$$

since $k \geq 3$ holds by the almost 3-connectedness of G. If $t \leq k - 1$, then Lemma 3.3 and (6) yield:

$$\tau \geq \frac{2}{n}. \tag{18}$$

(Note that (18) could be improved to $\tau \geq 1$ if G_0 is 3-connected.) If $t = k$ then almost 3-connectivity allows us to use the inequality (7) of lemma 3.3, which implies that in this case

$$\tau \geq 1. \tag{19}$$

Finally, (17)–(19) together with $n \geq 3$ yield:

$$2n\frac{b}{a} \geq \sum_{v \in S} \vartheta_v + \frac{1}{n}\pi \geq \frac{1}{n}\pi + \Big| \sum_{v \in S} \vartheta_v \Big| . \tag{20}$$

By the Cauchy-Schwarz inequality and the assumptions of the lemma we get:

$$\Big| \sum_{v \in S} \vartheta_v \Big| \leq \sum_{v \in S} |\vartheta_v| \leq \sqrt{s \sum_{v \in S} \vartheta_v^2} < \sqrt{n\mu(r)} \leq \sqrt{n\varepsilon} \leq \frac{1}{2n}\pi . \tag{21}$$

From (20) and (21) we easily get $a \leq \frac{4n^2}{\pi} b$ which implies (16). □

The last lemma shows that our algorithm works for arbitrary almost 3-connected graphs, and hence proves existence (cf. [10]) of a primal-dual circle packing (except for the circle at infinity) for a more general class of graphs than the 3-connected ones. Although we fixed the angles at the outer facial cycle $C = v_1 v_2 \dots v_k$ to be all equal to $\frac{k-2}{k}\pi$, we may choose for these angles any values α_i, $0 < \alpha_i < \pi$, whose total sum is $(k-2)\pi$; cf. [11, Chapter 2].

Theorem 3.9 *A plane graph G_0 admits a primal-dual circle packing representation in the plane with the circle corresponding to the unbounded face missing if and only if G_0 is a cycle graph. The angles α_i, $0 < \alpha_i < \pi$, whose total sum is $(k-2)\pi$, at the vertices v_i $(i = 1, \dots, k)$ of the outer face of the cycle graph G_0 can be chosen arbitrarily, and then the corresponding circle packing is unique up to a multiplicative factor and isometries of the plane.*

At the end, let us summarize the entire algorithm. We are given a 3-connected plane graph G_0 and the admissible error $\varepsilon > 0$. Construct the vertex-face graph $\tilde{G}$. Let ω be a vertex of degree 3 in $\tilde{G}$, and let $G = \tilde{G} - \omega$. (It is a simple consequence of Euler's formula that such a vertex always exists.) Then $k = 3$ and our goal is to find the radii $r = (r_v \mid v \in V(G))$ for the graph G such that there is a primal-dual circle packing of G_0 with radii $r^\circ = (r_v^\circ \mid v \in V)$ and for each vertex v of G we have $|r_v^\circ - r_v| \leq \varepsilon$. For this purpose we use Algorithm A described above. After computing the radii, one can determine coordinates of the centers of the circles in $\boldsymbol{R}^2$ as described below. Since $k = 3$, (5) implies that the angles at the three vertices of the vertex-face graph adjacent to ω are equal to $\pi/3$, and this implies that the radii at these vertices are all equal to each other. This shows that one can define also the circle corresponding to the vertex ω at infinity. Finally, Lemma 2.1 can be used to transform the obtained circle packing into a primal-dual circle packing where the circle at infinity corresponds to the unbounded face of G_0.

The centers P_v of circles C_v in a primal-dual circle packing with given radii r_v $(v \in V(G))$ can be computed as follows. Choose an arbitrary vertex

$v_0 \in V$ and put it in the origin of the plane. By using elementary geometry we can calculate the coordinates P_v for all vertices v that are adjacent to v_0 in G. We may think of this as tiling the neighborhood of v_0 by the basic quadrangles (cf. Figure 2) containing v_0. For each neighbor v of v_0 in G we repeat the process. Since one of the basic quadrangles containing v is already placed in the plane, other basic quadrangles have precisely one possibility to be placed around v. By repeating the procedure we exhaust the entire graph and obtain a primal-dual circle packing representation. The angle condition (4) and simple connectivity of the plane can be used to show that different ways of reaching the same basic quadrilateral Q yield the same position of Q in the plane. The reader is referred to [3, 10, 11] for additional details omitted in the above presentation.

4 The torus and the Klein bottle

In the case of closed surfaces of constant curvature 0, we may undertake the same way as described for the plane except that we do not need to treat a special vertex at the infinity. Now G is the whole vertex-face graph, and we use the same iteration procedure as described in Section 3 which finds appropriate radii satisfying (4). (The same procedure in the case of constant negative curvature is presented in details in [10].)

Vertices $x, y \in V(G_0)$ (with the possibility $x = y$) are said to be a *planar 2-separation* if there are internally disjoint simple paths π_1, π_2 from x to y on Σ such that:

(i) π_1, π_2 meet $G_0 \subset \Sigma$ only at their endpoints x, y.

(ii) The closed walk $\pi_1 \pi_2^{-1}$ bounds an open disk $D \subset \Sigma$.

(iii) D contains a vertex or a face of M_0.

The map M_0 is *reduced* if it contains no planar 2-separations. Maps with 3-connected graphs are reduced but we can have a reduced map whose graph is not 3-connected, or even not simple.

Theorem 4.1 (Mohar [10]) *A map on the torus or the Klein bottle admits a primal-dual circle packing representation if and only if it is reduced. The radii of such a representation are uniquely determined if we require that the minimum radius is equal to 1.*

A simple but interesting consequence of Theorem 4.1 is a characterization of maps that admit circle packings.

Corollary 4.2 *For a map M on the torus or the Klein bottle the following conditions are equivalent:*

(a) *M admits a circle packing representation.*

(b) *M admits a straight-line representation on a surface with constant curvature 0.*

(c) *M does not contain contractible loops or pairs of edges (possibly loops) with the same endpoint(s) that are homotopic relative their endpoint(s).*

To show equivalence of (a)–(c), one should note that by properly triangulating every face of a map satisfying (c), a reduced map is obtained. On the other hand, if a map does not satisfy (c), then it has no straight-line representation on a surface with constant curvature 0 by an easy application of the Gauss-Bonnet Theorem.

Theorem 4.1 shows the important role played by reduced maps. They have other characterizations as shown by the next result.

Proposition 4.3 ([10]) *Let $M_0 = (G_0, \Sigma)$ be a map on the torus or the Klein bottle. Then the following conditions are equivalent:*

(a) *The map M_0 is reduced.*

(b) *The graph of the universal cover of M_0 is 3-connected.*

(c) *The graph G_0 has no vertices of degree less than 3, no faces of size less than 3 and does not contain vertices x, y and two internally disjoint paths P_1, P_2 from x to y such that the closed walk $P_1 P_2^{-1}$ bounds a disk D on Σ and the only vertices on $P_1 \cup P_2$ that have a neighbor out of D are x and y.*

(d) *If there is a closed walk of length at most 4 in the vertex-face graph G that bounds a disk D in Σ, then D is a face of M.*

(e) *For every proper non-empty subset $S \subset V(G)$ of vertices of G we have:*

$$2|S| - |E(S)| \geq 1. \tag{22}$$

Since the property (d) of Proposition 4.3 is the same for M_0 as for its dual map M_0^*, equivalence of (a) and (d) shows:

Corollary 4.4 *The dual map M_0^* of M_0 is reduced if and only if M_0 is reduced.*

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