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# COHERENT FUNCTORS AND FAMILIES OF SPACE CURVES

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ABSTRACT. We give a summary of Auslander's theory of coherent functors and its application to the study of flat families in projective three-space. This is joint work with M. Martin-Deschamps and D. Perrin. Full details will appear in the paper [3], [4], [5], [6].

## 1 Functors on $A$-modules

Let $A$ be a commutative noetherian ring. Let Mod(A) be the category of all $A$-modules, and let $\mathcal{M} = \mathrm{Mod}_{fg}\,(A)$ be the category of finitely generated $A$-modules.

One often has to deal with an *$A$-linear functor $F$* from $\mathcal{M}$ to $\mathcal{M}$. Such a functor associates to each module $M \in \mathcal{M}$ another module $F(M) \in \mathcal{M}$; to a morphism $f : M \longrightarrow M'$ it associates a morphism $F(f) : F(M) \longrightarrow F(M')$; and to say that $F$ is $A$-linear means that the induced map

$$\mathrm{Hom}_A(M, M') \longrightarrow \mathrm{Hom}_A(F(M)), F(M'))$$

is $A$-linear.

Here are a few examples of $A$-linear functors.

1) For a fixed $M \in \mathcal{M}$, we define the functor $h_M$ by $h_M(N) = Hom_A\,(M, N)$ for all $N \in M$.

2) For a fixed $M \in \mathcal{M}$, the tensor product functor $N \mapsto M \otimes_A N$, which we denote by $M \otimes \cdot$.

3) If $A$ is an integral domain, we can define a functor $\tau$ by $\tau(M)$ = the torsion submodule of $M$.

4) The derived functors of 1) and 2), which are $\mathrm{Ext}^i_A(M, \cdot)$ and $\mathrm{Tor}^A_i(M, \cdot)$.

5) An example from algebraic geometry which is important in the sequel, is as follows. Let $Y = \mathrm{Spec}\ A$, and left $f : X \longrightarrow Y$ be a proper morphism of schemes. Let $\mathcal{F}$ be a coherent sheaf on X. Then we consider the functor $T^i$, for any $i \geq 0$, defined by

$$T^i(N) = H^i(X, \mathcal{F} \otimes_A N).$$

Note that the cohomology groups of a sheaf on $X$ have a natural structure of $A$-module, and these will be finitely generated over $A$ because $X$ is proper over $Y$ and $\mathcal{F}$ is coherent.

If $F$ is an A-linear functor form $\mathcal{M}$ to $\mathcal{M}$, and if

$$0 \to N' \to N \to N'' \to 0$$

is a short exact sequence of $A$-modules, then we obtain a sequence of $A$-modules

$$0 \to F(N') \to F(N) \to F(N'') \to 0.$$

If it exact in the middle, we say $F$ is *half exact*. If it is exact in the middle and on the left, $F$ is *left exact*. Similarly *right exact*. If exact everywhere, then $F$ is *exact*.

For example, in the list above, 1) and 3) are left exact; 2) is right exact; the functors in 4) are all half exact; and if $\mathcal{F}$ is flat over $Y$, then the functors in 5) will also be exact.

## 2 Coherent functors

Let us denote by Funct ($\mathcal{M}$) the set of all $A$-linear functors form $\mathcal{M}$ to $\mathcal{M}$. A morphism of functors $F \to G$ is a collection of maps $F(M) \to G(M)$ for each $M \in \mathcal{M}$ which commute with the induced maps $F(M) \to F(M')$ and $G(M) \to G(M')$ for any morphism of modules $M \to M'$. Given a morphism of functors $f : F \to G$, we can define new functors ker $f$, im $f$, coker $f$ by

$$(\ker f)\ (M) = \ker(F(M) \to G(M))$$

and similarly for im and coker. In this way Funct ($\mathcal{M}$) becomes an abelian category.

Among all these functors, some are better than others. Following Auslander [1], we define a functor F to be *coherent* if there are modules $M, N \in \mathcal{M}$ and an exact sequence of functors

$$h_M \to h_N \to F \to 0.$$

Then one can show that the set of all coherent functors $C$ forms an abelian subcategory of Funct ($\mathcal{M}$). In particular, if $f : F \to G$ is a morphism of coherent functors, then ker $f$, im $f$, and coker $f$ are also coherent. Also an extension of coherent functors is coherent.

In the examples above, the functors $h_M$ are coherent by definition. To see that the tensor product functor $M \otimes \cdot$ is coherent, let $P_1 \to P_0 \to M \to 0$ be a projective resolution of $M$. Then we get an exact sequence of functors

$$P_1 \otimes \cdot \to P_0 \otimes \cdot \to M \otimes \cdot \to 0.$$

For a finitely generated projective $A$-module $P$, let $P^\vee = \mathrm{Hom}_A(P, A)$ be its dual. Then the functor $P \otimes \cdot$ is isomorphic to $h_{P^\vee}$. Thus we see that $M \otimes \cdot$ is coherent. Note at this point it is essential that we are working with finitely generated modules over a noetherian ring $A$.

This argument shows more generally that if $P.$ is a complex of finitely generated projective $A$-modules, then the functors $h_i(P. \otimes \cdot)$ will be coherent. Thus we see that the functors $\mathrm{Ext}^i(M, \cdot)$ and $\mathrm{Tor}^i(M, \cdot)$ are coherent, by using a projective resolution of $M$ .

On the other hand, if A is an integral domain which is not a field, the torsion-submodule functor $\tau$ is not coherent. In fact it is not even finitely generated, meaning there is no M ∈ M admitting a surjective morphism $h_M \to \tau \to 0$.

With regard to the cohomology functors $T^i$ in example 5) above, if we assume that $X$ is projective over $Y$, and the sheaf $\mathcal{F}$ is flat over $Y$, then there exists a complex $L^\cdot$ of free finitely generated $A$-modules such that

$$T^i(N) = h^i(L^\cdot \otimes_A N)$$

for each $N \in \mathcal{M}$ [2, III.12.3]. Thus as above we see that the functors $T^i$ are coherent. But if $\mathcal{F}$ is not flat, the functors $T^i$ need not be coherent [3, 2.11].

## 3 Duality

For a finite dimensional vector space $V$ over a field $k$, we are familiar with the dual vector space $V^* = \mathrm{Hom}_k\,(V, k)$. The operation of taking the dual vector space is an exact, contravariant functor $*$ from the category of vector spaces to itself, with a natural isomorphism $** = \mathrm{id}$.

For finitely generated projective modules $P$ over a ring, the operation of taking the dual module $P^\vee = \mathrm{Hom}\,(P, A)$ has similar properties, but this operation does not extend to the category of all A-modules. However if we consider the larger category $C$ of coherent functors (here I am thinking of $\mathcal{M}$ as being embedded in $C$ by associating to the module $M$ the functor $M \otimes \cdot$) there is a good notion of duality.

Given a coherent functor $F \in C$, represent it as a cokernel

$$h_M \to h_N \to F \to 0.$$

The map of functors $h_M \to h_N$ arises from a map of modules $f : N \to M$, so we can define the dual functor $F^*$ by

$$F^* = \ker\,(N \otimes \cdot \to M \otimes \cdot).$$

Then one shows that this is well-defined (i.e. independent of the choice of the representation $h_M \to h_N \to F$), and that $*$ is a contravariant exact functor from $C$ to $C$ with $** \cong \mathrm{id}_C$. In particular, the functors $h_M$ and $M \otimes \cdot$ are dual to each other. One can also see easily that $\mathrm{Ext}^i(M, \cdot)$ and $\mathrm{Tor}_i(M, \cdot)$ are dual coherent functors. The duality interchanges left exact and right exact functors, and sends half exact functors into half exact functors. The exact functors are all of the form $P \otimes \cdot$, where $P$ is a finitely generated projective $A$-module. The dual of $P \otimes \cdot$ is $h_P \cong P^\vee \otimes \cdot$, so the duality $*$ on $C$ extends the duality $\vee$ on projective $A$-modules.
Using this notion of dual coherent functors, one can express Grothendieck's duality theorem (which generalizes Serre duality over a field) in a particularly nice way.

**Theorem 3.1** *[3, 7.4] Let $Y = Spec A$ , let $X$ be a smooth projective scheme over $Y$ of relative dimension n, and let $\omega = \Omega^n_{X/Y}$ be the relative dualizing sheaf. Then for any coherent sheaf $\mathcal{F}$ on $X$, flat over $Y$, the functors*

$$Ext^{n-i}_X(\mathcal{F}, \omega \otimes_A \cdot) \text{ and } H^i(X, \mathcal{F} \otimes_A \cdot)$$

*are dual coherent functors.*

The value of this result is that while there is no simple relationship between the individual A-modules $\mathrm{Ext}^{n-i}_A(\mathcal{F}, \omega)$ and $H^i(X, \mathcal{F})$, the theorem gives a duality between the corresponding functors.

## 4 Space curves

Let $X$ be the projective space $\mathbb{P}^3_k$ over an algebraically closed field $k$. A *curve* in $X$ is a closed subscheme $C$ of pure dimension one with no embedded points. To the curve $C$ we associate its *Rao module*

$$M_C = \bigoplus_{n \in \mathbb{Z}} H^1(X, \mathcal{I}_C(n)),$$

where $\mathcal{I}_C$ is the sheaf of ideals of the curve C. This module is a finite length graded module over the polynomial ring $R = k[x, y, z, w]$.

An *N-type resolution* of the curve C is an exact sequence

$$0 \to \mathcal{P} \to \mathcal{N} \to \mathcal{I}_C \to 0$$

on $X$, where $\mathcal{P}$ is *dissocié*, meaning isomorphic to a direct sum $\oplus \mathcal{O}_X(-n_i)$, and $\mathcal{N}$ is locally free with $H^2_*(\mathcal{N}) = 0$. Here $H^i_*(\mathcal{N})$ means

$$\oplus_{n \in \mathbb{Z}} H^i(X, \mathcal{N}(n)).$$

Two curves $C$ and $C'$ are linked if there exists a complete intersection curve $D$ containing $C$ and $C'$ and such that

$$\mathcal{I}_{C,D} \cong \mathcal{H}om(\mathcal{O}_{C'}, \mathcal{O}_D)$$

and

$$\mathcal{I}_{C',D} \cong \mathcal{H}om(\mathcal{O}_{C'}, \mathcal{O}_D)$$

Two curves $C$ and $C'$ are *equivalent up to biliason* if they can be connected by a chain of an even number of such linkages.

Now we can state the classical Rao theorem as follows.

**Theorem 4.1** *[8],[9] Let $X = \mathbb{P}^3_k$. Then there are one-to-one correspondences between the following three sets:*

*i) the set of curves $C \subseteq X$,up to biliaison equivalence*

*ii) the set of locally free sheaves $\mathcal{N}$ on $X$ satisfying $H^2_*(\mathcal{N}) = 0$, up to stable equivalence (adding dissocié sheaves) and twists*

*iii) the set of finite length graded $R$-modules, modulo isomorphism, up to shift in degrees.*

The correspondences are given by associating to a curve $C$ its Rao module $M_C$, and the sheaf $\mathcal{N}$ coming from an $N$-type resolution.

Furthermore, each biliaison equivalence class of curves (except for the class of $ACM$ curves, which corresponds to the 0 module over $R$) satisfies the Lazarsfeld-Rao property: in each biliaison equivalence class there is a minimal curve $C_0$, and any other curve $C$ in the biliaison class can be obtained from $C_0$ by a finite number of elementary biliaisons, followed by a deformation with constant cohomology (see [7] for a more detailed statement).

## 5 Families of space curves

Our purpose here is to find the analogue of the results of § 4 for families of space curves.

Let $A$ be a noetherian local ring, let $Y = \operatorname{Spec} A$, and let $X = \mathbb{P}^3_Y$. A curve in X is a closed subscheme $C \subseteq X$, flat over Y, with the property that for each $t \in Y$, the fibre $C_t \subseteq X_t = \mathbb{P}^3_{k(t)}$ is a curve in the previous sense, namely pure dimension 1 with no embedded points.

Liaison and biliaison are defined as before.

For the $N$-type resolution, we require only that there be an exact sequence

$$0 \to \mathcal{P} \to \mathcal{N} \to \mathcal{I}_C \to 0$$

with $\mathcal{P}$ dissocié and $\mathcal{N}$ locally free on $X$.

If $\mathcal{N}$ and $\mathcal{N}'$ are two locally free sheaves on X, we say that a morphism $f : \mathcal{N} \to \mathcal{N}'$ is a *pseudo-isomorphism* (psi for short) if it induces

1) an isomorphism of functors $H^1_*(\mathcal{N} \otimes_A \cdot) \to H^1_*(\mathcal{N}' \otimes \cdot)$ and

2) an injection of functors $H^2_*(\mathcal{N} \otimes_A \cdot) \to H^2_*(\mathcal{N}' \otimes \cdot)$.

We say two locally free sheaves $\mathcal{N}$ and $\mathcal{N}'$ are *equivalent for psi* if they can be joined by a chain of sheaves which admit psi between them in one direction or the other.

**Theorem 5.1** *[4] Let $X = \mathbb{P}^3_Y$, with $Y = Spec\ A$ as above. Any curve $C \subseteq X$ admits an N-type resolution. The corresponding locally free sheaf $\mathcal{N}$ is uniquely determined up to psi. If $C$ and $C'$ are two curves, with sheaves $\mathcal{N}$, $\mathcal{N}'$ in their N-type resolutions, then $C$ and $C'$ are equivalent for biliaison if and only if $\mathcal{N}$ and $\mathcal{N}'$ are equivalent for psi.*

**Theorem 5.2** *[5] With $X$, $A$ as above, if $\mathcal{N}$ is a locally free sheaf on $X$, there exists a curve $C_0 \subseteq X$ of minimal degree with $N$-type resolution $\mathcal{N}_0$ which is psi to $\mathcal{N}$, and any other curve $C'$ in the same biliaison equivalence class can be obtained from $C_0$ by a finite number of elementary biliaisons followed by a deformation with constant cohomology.*

For our third result, we need an analogue of the Rao module of a curve over a field. The natural choice would seem to be the *Rao functor*

$$F_C = \bigoplus_{n \in \mathbb{Z}} H^1(X, \mathcal{I}_C(n) \otimes_A \cdot)$$

This is a functor from $A$-modules to graded $R_A = A[x,y,z,w]$-modules. It is a direct sum of its graded pieces $F_n$, each of which is a coherent functor on $A$-modules.

One can show easily enough that curves $C$ and $C'$ which are equivalent up to biliaison give rise to functors $F_C$ and $F_{C'}$ which differ only by a shift in degrees. However, the converse statement is false, so we need a more refined notion to play the role of the Rao module.

As above, let $A$ be a noetherian local ring, and let $R_A = A[x,y,z,w]$. A *triad* is a complex $L^0 \to L^1 \to L^2$ of finitely generated graded $R_A$-modules, flat over $A$, and such that $h^i(L^\cdot)$ is a finitely generated $A$-module for $i = 1,2$. A morphism of triads is simply a morphism of complexes. A *morphism* of triads $f : L^\cdot \to L'^\cdot$ is a *pseudo-isomorphism* (psi for short) if it induces

1) an isomorphism of functors $h^1(L^\cdot \otimes_A \cdot) \to h^2(L'^\cdot \otimes_A \cdot)$ and

2) an injection of functors $h^2(L^\cdot \otimes_A \cdot) \to h^2(L'^\cdot \otimes_A \cdot)$.

We say two triads are *equivalent* for psi if they can be joined by a finite chain of such morphism, in either direction.

To a triad $L$ we associate the sheaf

$$\mathcal{N} = ker(\tilde{L^0} \to \tilde{L^1})$$

which will be locally free on $X$. We say that a triad $L^\cdot$ is associated to a curve $C$ if $\mathcal{N}$ is psi equivalent to a sheaf coming from an $N$-type resolution of $C$. In that case the Rao functor of $C$ can be obtained as $h^1(L^\cdot \otimes \cdot)$.

**Theorem 5.3** *[6] Let $C$ and $C'$ be curves with associated triads $L^\cdot$ and $L'^\cdot$. Then $C$ and $C'$ are equivalent for biliaison if and only if $L^\cdot$ and $L'^\cdot$ are equivalent for psi.*

In the special case of a discrete valuation ring $A$, with residue field $k$ and quotient field $K$, suppose given finite length graded modules $M_0$ over $k[x,y,z,w]$ and $M_1$ over $K[x,y,z,w]$. Then we can compute effectively all possible triads $L^\cdot$ whose associated modules are $h^1(L^\cdot \otimes k) = M_0$ and $h^1(L^\cdot \otimes K) = M_1$, and thus in principle we can determine all possible flat families of curves $C \subseteq \mathbb{P}^3_A$ whose special and general fibre $C_0$ and $C_1$ are in the biliaison equivalence classes determined by the Rao modules $M_0$ and $M_1$. While the computations quickly become quite complicated, we are still hopeful that this will be a fruitful method for studying properties of the Hilbert scheme of curves in $\mathbb{P}^3$ over a field $k$.

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