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**NONLINEAR PROCESSES OF INJECTION  
OF LIQUIDS THROUGH POROUS MATERIALS\***

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ABSTRACT. We review some problem in which the injection of a liquid through a porous medium is coupled with phenomena affecting the rheological properties of system. Among the many processes of this type we illustrate with some details various flows occurring in the manufacturing of composite materials and a model for the penetration of water in a porous medium containing hydrophile granules.

## **1 Introduction**

We suppose that the reader is familiar with the basic notions of porous media (very classical references are [2] [3] and the more recent monograph [1]).

We just recall that

1. the *porosity*  $\varepsilon$  of medium is the volume fraction available for the flow (i.e. the volume of the connected *pores*),
2. the saturation  $S$  is the fraction of the pore volume occupied by a liquid present in the medium ( $S \equiv 1$  corresponding to the saturation regime),
3. the basic law governing the flow of a liquid through a porous medium is the *Darcy's law*

$$\mathbf{q} = -k(S, \varepsilon)\nabla p \quad (1)$$

(in which gravity has been neglected, since this will be the case in all the problems we are going to consider), where  $\mathbf{q}$  is the *volumetric velocity* ( $\mathbf{q} \cdot \mathbf{n}$  = volume of liquid crossing the unit surface with unit normal vector  $\mathbf{n}$  per

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unit time),  $k$  is the *hydraulic conductivity* (the ratio between so-called permeability, of the medium and the viscosity of the flowing liquid),  $p$  is the pressure.

The quantities  $S$  and  $p$  are related to each other because of capillarity:  $S = 1$  for  $p < p_s$  (saturation pressure) and is otherwise an increasing function of  $p$ .

If  $\varepsilon$  is constant and the flow is saturated, then  $\kappa = \text{constant}$  and incompressibility yield  $\text{div } \mathbf{q} = 0$ , so that  $p$  is a harmonic function in the flow domain.

All injection problems through initially dry media are free boundary problems, since the motion of the advancing front of the liquid is not known a-priori.

Therefore such problems are nonlinear even in the simplest situations. In many technological applications we have to deal with injection problems in which the flow produces significant changes in the system (either of mechanical or chemical or even thermal nature), which in turn affect the rheological properties (the porosity, the permeability, the viscosity, ecc.)

In such cases Darcy's law must be coupled with some kinetic equation or a diffusion/transport equation, etc. The resulting problems are generally very difficult and in most cases they are partly or completely open.

Many classes of technological (or natural) processes belong to this category.

Injection problems have been studied quite intensively in the past in various situations. It would be quite beyond the scope of this presentation to review such a large literature. Let us just mention the papers [27], [15], [16], [13], [19], [18] (although there are many more) and the text of the talk given by M. Primicerio in this same series some years ago [24].

A remarkable (and somehow curious) example is the so-called espresso-coffee problem (hot water injected through packed ground coffee under high pressure gradients), which is matematically very rich (with various question still open). We refer to [10] for a survey and for the main references.

A natural processes (with suction instead of injection) is related to saturated ground freezing, in which a complex thermodynamical mechanism produces and upward flow of water from the confining water table, giving rise the unusual and spectacular (but also dangerous) phenomena.

An excellent review of ground freezing is [26].

The injection of a viscous liquid through the gap between two parallel plates very close to each other, the so-called Hele-Shaw problem, is formally very similar to a saturated flow in a porous medium. Therefore we can find examples also in this field (for instance non-isothermal Hele-Shaw flow [11]).

For the necessity of being coincide we will not deal with any of the problems above.

Instead we have selected to talk about two main classes of injection problems. The first class consist of problems describing various stages of the manufacturing of composite materials (porous media filled by a polymerized resin). The second class deals with water injection in a porous material provided with hydrophile granules (babies diapers). We will follow the model presented in [9], [8] (an entirely different approach has been previously studied in [28], [22]). All the problems considered are set in the one-dimensional geometry. An important reference for what we are going to say is the large review paper by L. Preziosi [23] on deformable porous media.

Our exposition will not be exhaustive, not only for obvious reasons of conciseness, but also because these studies are more at the beginning than at the conclusion. There is still a lot to do about modeling, numerical calculations, theoretical investigation of the mathematical structure of the problems. This area is fast developing and is an incredible source of interesting mathematical problems.

Before we proceed to illustrating complicated injection problems, it is appropriate to recall very briefly the basic model problem, the so-called Green-Ampt model [14], which dates back to 1911. We will do this in the next section.

## 2 The Green-Ampt model

Well known to soil engineers, the Green-Ampt model [14] describes (very schematically) the process of injection of a liquid through a porous medium in the simple case in which the geometric and rheological properties of the medium are not affected by the flow (porosity, permeability, fluid viscosity and hence hydraulic conductivity are all constants). In addition the geometry is planar: injection occurs through a planar surface into a semi-infinite medium, with a uniformly distributed injection pressure  $p_0(t)$ .

Another simplification introduced into the model is the extrapolation from the usual unsaturated/saturated flow (due to the presence of a capillary fringe) to the extreme case in which the medium is either saturated or dry (this is however a good approximation in fast penetration processes).

In such a situation, as we said, pressure is a harmonic, i.e. linear function of the space coordinate  $x$  in the direction of the flow.

Denoting by  $x = 0$  the inflow surface and by  $x = s(t) > 0$  the penetration front, on which we set  $p = 0$ , we have

$$p(x, t) = p_0(t) \frac{s(t) - x}{s(t)} \quad (2)$$

Of course the function  $s(t)$  is unknown and we need some additional information, which is provided by the fact that the penetration front is a material surface and therefore it must obey Darcy's law, which consistently

with the rest of the paper we write in the form (1), neglecting gravity. Since the porous medium is supposed to be homogeneous and not modified by the flow, in (1) we take  $k = \text{constant}$ , which allows us to derive the simple expression (2).

If  $\varepsilon$  is the medium porosity, in order to provide the volumetric velocity  $q$  the water molecules must travel with the speed  $\frac{q}{\varepsilon}$  (since  $\varepsilon$  is also the void ratio over a cross section). This amounts to saying that

$$\dot{s}(t) = -\frac{k}{\varepsilon} \frac{\partial x}{\partial \lambda} = -\frac{k}{\varepsilon} \frac{p_0(t)}{s(t)}. \quad (3)$$

If the medium is initially dry ( $s(0) = 0$ ) we obtain

$$s(t) = \left[ 2 \frac{k}{\varepsilon} \int_0^t p_0(\tau) d\tau \right]^{1/2}, \quad (4)$$

which is the desired solution.

As we shall see, slight changes to this very simple model, lead to extremely difficult complications.

A first application of the model above to composite material manufacturing can be found in [7]. The process consists in squeezing an array of alternate layers of (artificial) porous media and liquid resin, so that the resin penetrates the porous layers. The compression is performed in an autoclave. In its simplest setting the problem can be considered as a superposition of Green-Ampt processes. However, for reasons that will be clear later, the temperature can vary during the process.

Therefore it is of great importance to analyze non-isothermal injection processes.

### 3 Non-isothermal injection processes

It is well known that the viscosity of any liquid depends in a very sensitive way on temperature. As a consequence, the hydraulic conductivity for a flow in a porous medium is also temperature dependent.

In addition, when heat is transported in a medium which is only partially occupied by a liquid, the thermal properties of the system are different in the wet and in the dry region. Neglecting the heat capacity and thermal conductivity of air, we can say that the heat capacity of the system is

$$c = (1 - \varepsilon)c_s + \varepsilon c_l \chi_w, \quad (5)$$

where  $c_s$  is the heat capacity of the solid grains,  $c_l$  the heat capacity of the liquid, and  $\chi_w$  the characteristics function of the wet region. In (5)  $\varepsilon$  is the porosity, and we continue to assume that there is an interface separating the non-penetrated region from a saturated region.

Similarly, for the thermal conductivity we have

$$\lambda = (1 - \varepsilon)\lambda_s + \varepsilon\lambda_\ell\chi_w \quad (6)$$

keeping the one-dimensional geometry, we can summarize the heat diffusion convection equation by

$$c \frac{\partial u}{\partial t} - \frac{\partial}{\partial x} \left( \lambda \frac{\partial u}{\partial x} \right) + \dot{s}(t) \varepsilon c_\ell \chi_w \frac{\partial u}{\partial x} = 0, \quad (7)$$

to be solved in the wet region ( $0 < x < s(t), t > 0, \chi_w = 1$ ) and in the dry region ( $s(t) < x < l, t > 0, \chi_w = 0$ ).

At the interface  $x = s(t)$  we have to impose

(i) the continuity of temperature

$$u(s(t) - t) = u(s(t) +, t) \quad (8)$$

(ii) the continuity of diffusive heat flux

$$\left( \lambda \frac{\partial u}{\partial x} \right) \Big|_{x=s(t)-} = \left( \lambda \frac{\partial u}{\partial x} \right) \Big|_{x=s(t)+} \quad (9)$$

This kind of thermal problem is known as a diffraction problem.

In the scheme above the penetrated layer is the region  $0 < x < s(t), t > 0$ . At all points the liquid velocity is  $\dot{s}(t)$ , due to incompressibility.

Of course problem (7)-(9) must be complemented by the initial and boundary conditions, which will not be commented here for the sake of brevity.

The substantial difference between (7)-(9) and a classical thermal diffraction problem (see e.g. [20]) is the fact that the motion of the interface is unknown.

Now, since the interface is a material surface, it obeys Darcy's law (1), in which the coefficient  $\kappa$  depends on the temperature  $u$ . We rewrite (1) in the form

$$R(u) \dot{s}(t) = - \frac{\partial p}{\partial x} \quad (10)$$

where  $R = \frac{1}{\kappa}$  is the *hydraulic resistivity*.

In this way we can take into account the boundary conditions for pressure:  $p = p_0 > 0$  (e.g. constant) at the inflow surface  $x = 0, p = 0$  on the penetration front. Just integrate (10) w.r.t.  $x$  over the wet region:

$$\dot{s}(t) = \frac{p_0}{2} \left\{ \int_0^{s(t)} R(u(x, t)) dx \right\}^{-1}. \quad (11)$$

Together with the initial condition  $s(0) = s_0$  the latter equation completes the model.

Existence and uniqueness of the solution (in the classical sense) for the free boundary problem (7), (8), (9), (11) with conditions

$$u(0, t) = u_1(t), \quad u(\ell, t) = u_2(t), \quad t > 0 \quad (12)$$

$$u(x, 0) = u_0(x), \quad 0 < x < \ell \quad (13)$$

$$s(0) = b \in (0, \ell), \quad (14)$$

have been proved in [4] under suitable assumptions on the data (the case  $b = 0$  is treated separately).

The demonstration technique uses a fixed point argument based on the following steps:

1) Define the set

$$S = \left\{ s \in C([0, T]) : s(0) = b, |s|_y \leq M \right\} \quad (15)$$

where  $y$  is chosen arbitrarily in  $(1/2, 1)$ ,  $|\cdot|_y$  is the corresponding Hölder norm,  $M$  and  $T$  have to be selected later.

2) For any  $s \in S$  solve the thermal diffraction problem (7), (8), (9), (12), (13). The determination of corresponding temperature  $u(x, t)$  is reduced by means of classical tools to the solution of a system of four integral equations of Volterra type with weakly singular kernels.

3) The function  $u(x, t)$  is used to define the mapping  $\mathcal{T} : s \rightarrow \sigma$  in the following way

$$\dot{\sigma}(t) = \frac{p_0}{2} \left( \int_0^{s(t)} R(u(xt)) dx \right)^{-1}, \quad \sigma(0) = b \quad (16)$$

so that a fixed point of  $\mathcal{T}$  corresponds to a solution of the original free boundary problem.

4) The constants in the set (15) are chosen in such a way that  $\sigma \in S$  and the continuity of the mapping  $\mathcal{T}$  in the topology of  $C([0, T])$  is proved. Schauder's theorem guarantees the existence of at least one fixed point.

5) A finer analysis in the  $C^1([0, T])$  topology shows that  $\mathcal{T}$  is in fact a contraction in a sufficiently small time interval and uniqueness follows.

We are now ready to consider a more realistic model for resin injection.

#### 4 Resin injection with curing

When the penetrating resin undergoes a polymerization process (curing), the model above must be modified in order to take into account

1. the polymerization kinetics
2. the heat released during polymerization

3. the dependence of viscosity on the degree of polymerization.

The experimental evidence of these facts is reported e.g. in [17], [21]. The following model has been developed in [5].

The additional quantity to be introduced is the so-called curing degree  $\alpha \in (0, 1)$ : for  $\alpha = 0$  the resin is in the monomer state, for  $\alpha = 1$  it is fully polymerized. Of course the medium resistivity depends very much on  $\alpha$  as well as on temperature.

Now all the thermal coefficients in the penetrated region depend on  $\alpha$ :

$$c = (1 - \varepsilon)c_s + \varepsilon [\alpha c_p + (1 - \alpha)c_\ell], \quad (17)$$

where  $c_p, c_\ell$  refer to the polymerized and to the non-polymerized component respectively. A similar convex combination gives the thermal conductivity

$$\lambda = (1 - \varepsilon)\lambda_s + \varepsilon [\alpha\lambda_p + (1 - \alpha)\lambda_\ell]. \quad (18)$$

Neglecting the volume change accompanying polymerization, we still have that the liquid moves at the velocity  $\dot{s}(t)$ , and the curing kinetics can be written as

$$\frac{\partial \alpha}{\partial t} + \dot{s}(t) \frac{\partial \alpha}{\partial x} = \mu(u, \alpha) \quad (19)$$

where  $\mu$  is an experimental function. This equation requires a boundary condition

$$\alpha(0, t) = 0 \quad (20)$$

(non-polymerized liquid is injected), and, if  $s(0) = b > 0$ , the initial condition

$$\alpha(x, 0) = 0, \quad 0 < x < b. \quad (21)$$

Finally, the thermal balance equation modifies to

$$c \frac{\partial u}{\partial t} - \frac{\partial}{\partial x} \left( \lambda \frac{\partial u}{\partial x} \right) + \dot{s}(t) \varepsilon [\alpha c_p + (1 - \alpha)c_\ell] \mathcal{X}_w \frac{\partial u}{\partial x} = \varphi \mu(u, \alpha) \mathcal{X}_w \quad (22)$$

where  $\varphi$  is a positive constant.

A slightly simplified version of this problem has been considered in [5]: zero temperature boundary data and r.h.s. of (22) replaced by a given function  $h(x, t)$ . A weak formulation is then introduced for the thermal balance equation:

$$\int_0^T \int_0^\ell \{-cuv_t + \lambda u_x v_x - \dot{s} [\alpha c_p + (1 - \alpha)c_\ell] \mathcal{X}_w uv_x - ruv\} dx dt$$

$$+ \int_0^1 u_0(c v)|_{t=0} dx = \int_0^T \int_0^\ell h v dx dt, \quad (23)$$

where  $u \in L^2(0, T; H_0^1(\Omega))$ , for any  $v \in L^2(0, T; H_0^1(\Omega))$ ,  $v(\cdot, T) = 0$ ,  $v_t \in L^2(Q)$ ,  $\Omega$  being the interval  $(0, l)$  and  $Q = (0, l) \times (0, T)$ . In equation (4.8) the coefficient  $r$  is defined as follows

$$r = c_t + \dot{s} \varepsilon [\alpha c_p + (1 - \alpha) c_l] X_w = \varepsilon (c_p - c_l) \mu(u, \alpha) X_w, \quad (24)$$

where (19) has been used.

Again a fixed point argument is used:

- 1) We select a set of  $C^1$  functions in which we prescribe the curve  $x = s(t)$  and a set of Lipschitz continuous functions in which prescribe  $\alpha(x, t)$ .
- 2) We use the existence and uniqueness results known in the literature [20] to find the solution  $u(x, t)$  of (23) and we observe that  $u$  is continuous in  $x$  for a.a.  $t$  and  $\langle \cdot, t \rangle u_x(\cdot, t) \in H^1(\Omega)$  for a.a.  $t$ , so that the diffraction conditions are valid in an almost classical sense.
- 3) We introduce a mollified sequence  $\alpha^\sigma$  of  $\alpha$ , defined throughout  $Q$  and we consider the corresponding classical version of (23). Estimates of  $\|u(\cdot, t)\|_{L^2(\Omega)}$ ,  $\|u_x\|_{L^2(Q)}$  independent of  $\sigma$  are obtained by means of standard arguments. Much more difficult are the estimates on  $\|u_x(\cdot, t)\|_{L^2(\Omega)}$ , and on  $\|u_t\|_{L^2(Q)}$ , obtained in [5].
- 4) Using the estimates above it is possible to prove that if a sequence  $\{s_n\}$  converges in  $C^1([0, T])$  and a sequence  $\{\alpha_n\}$  of uniformly Lipschitz continuous functions converges uniformly in  $\overline{Q}$ , then the corresponding sequence  $\{u_n\}$  of solutions of (23) converges uniformly in  $\overline{Q}$ .
- 5) On the basis of the preceding results, it can be proved that the operator  $\mathcal{F} : (\alpha, s) \rightarrow (\tilde{\alpha}, \tilde{s})$  defined by

$$\frac{d \tilde{s}}{dt} = \frac{p_0}{\varepsilon} \left( \int_0^{s(t)} R(\alpha, u) dx \right)^{-1}, \quad \tilde{s}(0) = b, \quad (25)$$

$$\frac{\partial \tilde{\alpha}}{\partial t} + \frac{d \tilde{s}}{dt} \frac{\partial \tilde{\alpha}}{\partial x} = \mu(\tilde{\alpha}, u), \quad (26)$$

$$\alpha(x, 0) = 0, \quad \alpha(0, t) = 0 \quad (27)$$

has a fixed point.

It must be remarked that this procedure requires uniform estimates of  $\left\| \frac{\partial \tilde{\alpha}}{\partial x} \right\|_{L^\infty(Q)}$ ,  $\left\| \frac{\partial \tilde{\alpha}}{\partial t} \right\|_{L^\infty(Q)}$  which are quite delicate. For the details see [5]. Further extensions of the model are under investigation.



## 5 Injection processes with compression and relaxation

In the preceding two sections we have considered the complications arising in the standard Green-Ampt problem due to the presence of non homogeneous thermal fields and the possible occurrence of curing. However, deformation of the porous matrix (due to the high injection pressure) may also be important. Starting from the analysis of the paper [25], the following model has been studied by L. Billi and A. Farina [6].

Let us consider a one-dimensional problem in which the applications of the injection pressure at time  $t = 0$  produces an instantaneous shrinking of the porous material. So in the initial situation the liquid has not yet penetrated and the porous medium is compressed to a minimum porosity  $\theta_c$ . When the liquid penetrates, the porous matrix undergoes a relaxation process with the porosity increasing to a higher value  $\theta_r$ . Therefore we have to describe two processes, namely

1. the liquid flow in the (relaxing) porous material,
2. the relaxation of the porous matrix.

Also we have two free surfaces:

(a) the penetration front, i.e. the free surface of the liquid flow,

(b) the relaxation front, i.e. the free surface of the motion of the solid component.

The basic quantities are the following:

- $u_l$ , the liquid (molecular) velocity
- $u_s$ , the solid velocity
- $\theta$ , the porosity of the penetrated material
- $p$ , the pressure

The governing equations are

(I) DARCY'S LAW (velocity of liquid relative to solid)

$$u_l - u_s = - \frac{K(\theta)}{\mu\theta} \frac{\partial p}{\partial x} \quad (28)$$

(II) LIQUID MASS BALANCE

$$\frac{\partial \theta}{\partial t} + \frac{\partial}{\partial x} (\theta u_l) = 0 \quad (29)$$

(III) SOLID MASS BALANCE

$$\frac{\partial(1 - \theta)}{\partial t} + \frac{\partial}{\partial x} [(1 - \theta)u_s] = 0 \quad (30)$$

(IV) MOMENTUM BALANCE

$$\frac{\partial p}{\partial x} + \frac{\partial \sigma(\theta)}{\partial x} = 0 \quad (31)$$

In (28) the conductivity  $K(\theta)$  is an experimental function (typically  $K'(\theta) > 0, K(0) = K_0 > 0$ , e.g.)  $K(\theta) = K_0 e^{-\alpha(1-\theta)}$ . In (31) inertia forces are neglected,  $\sigma(\theta)$  represents the (unidirectional) stress on the solid and is an experimental function such that  $\sigma'(\theta) < 0$ .

Introducing the compound velocity

$$v = \theta u_c + (1 - \theta) u_s \quad (32)$$

we see that

$$\frac{\partial v}{\partial x} = 0 \quad (33)$$

i.e. the overall motion is volume preserving and  $v = v(t)$ .

Combining in a suitable way (28), (31) and (32) we obtain the basic equation for the porosity  $\theta(x, t)$ :

$$\frac{\partial \theta}{\partial x} + \frac{\partial}{\partial x} \left[ (1 - \theta) \frac{K(\theta)}{\mu} \sigma'(\theta) \frac{\partial \theta}{\partial x} \right] + v(t) \frac{\partial \theta}{\partial x} = 0, \quad (34)$$

to be satisfied in the penetrated region. Note that (34) is a quasilinear (forward) uniformly parabolic equation: recall that  $\sigma' < 0, 0 < \theta_c \leq \theta \leq \theta_r < 1$ , and assume all the regularity needed for  $K(\theta), \sigma(\theta)$ . Remark also that the coefficient  $v(t)$  in (34) is still unknown.

In order to complete the mathematical model, not only we have to find  $v(t)$ , but also we must specify the conditions on the two free boundaries:  $x = x_w(t)$ , the penetration front,  $x = x_r(t)$ , the relaxation front.

It is easy to realize that  $v(t)$  is nothing but the injection volumetric velocity (apply (32) on the injection surface on the side of the pure liquid region). The injection (volumetric) velocity coincides with the volumetric velocity calculated on the penetration front

$$v(t) = \frac{K(\theta_c)}{\mu} \sigma'(\theta_c) \frac{\partial \theta}{\partial x} \Big|_{x=x_w(t)} \quad (35)$$

The conditions prescribed at the penetration front are

$$\theta(x_w(t), t) = \theta_c \quad (36)$$

$$\dot{x}_w(t) = \frac{K(\theta_c)}{\mu \theta_c} \sigma'(\theta_c) \frac{\partial \theta}{\partial x} \Big|_{x=x_w(t)} \quad (37)$$

(the molecular penetration velocity).

The conditions on the relaxation front are

$$\theta(x_r(t), t) = \theta_r, \quad (38)$$

$$\dot{x}_r(t) = \frac{K(\theta_c)}{\mu\theta_c} \sigma'(\theta_c) \frac{\partial \theta}{\partial x} \Big|_{x=x_w(t)} - \frac{K(\theta_r)}{\mu\theta_r} \sigma'(\theta_r) \frac{\partial \theta}{\partial x} \Big|_{x=x_r(t)} \quad (39)$$

in other words  $\dot{x}_r(t) = u_s$  at  $x = x_r(t)$  and the latter quantity is calculated from (28) and (38), making use of the fact that  $u_l = \dot{x}_w(t)$ .

Thus the complete model consist of equations (34)-(39). Let us make some comments on the structure of this double free boundary problem.

First of all we note that once  $\theta(x, t)$  has been determined in the penetrated region, the pressure  $p(x, t)$  can be calculated integrating (31) over the penetrated layer. In particular

$$\sigma(\theta_r) - \sigma(\theta_c) = -P_0, \quad (40)$$

$P_0$  being the pressure difference between the two free boundaries, which is prescribed. Since  $\sigma(\theta_r) = 0$  (the relaxation surface is stress free) (40) provides a relationship between  $\theta_c$  and  $P_0$ , which is actually the way in which the pressure boundary conditions comes into play.

Mathematically, the pair (36), (37) coincides with the well known Stefan conditions for the one-phase solidification problem. On the other hand, condition (39) involves the derivate  $\frac{\partial \theta}{\partial x}$  evaluated at the both the boundaries.

Another peculiarity is the presence of the term  $v(t)$ , i.e of the expression (35) in the governing differential equation.

The tools applied in [12] can provide an existence theorem. The quoted paper [6] is devoted to the determination of self similar solutions. Choosing a Lagrangian coordinate  $\xi$  labelling the solid particles and introducing the so-called void ratio

$$e = \frac{\theta}{1 - \theta} \quad (41)$$

the above system is reduced to

$$\frac{\partial e}{\partial t} = \frac{\partial}{\partial \xi} \left[ F(e) \frac{\partial e}{\partial \xi} \right], \quad 0 < \xi < \xi_f(t), t > 0 \quad (42)$$

$$e(0, t) = e_r = \frac{\theta_r}{1 - \theta_r} \quad (43)$$

$$e(\xi_f(t), t) = e_c = \frac{\theta_c}{1 - \theta_c} \quad (44)$$

$$\dot{\xi}_f(t) = -\frac{F(\theta_c)}{e_c} \frac{\partial e}{\partial \xi} \Big|_{\xi=\xi_f(t)} \quad (45)$$

$$\xi_f(0) = 0, \quad (46)$$

where  $F(e)$  is easily determined.

Defining

$$\chi = \frac{\xi}{\xi_f(t)}, \quad (47)$$

we look for a solution to (42)-(46) in the form

$$e(\xi, t) = u(\chi), \quad 0 < \chi < 1, \quad (48)$$

and because of (45)  $\xi_f(t)$  must then satisfy

$$\xi_f \dot{\xi}_f = -\frac{F(e_c)}{e_c} u'(1), \quad (49)$$

so that the penetration front in the Lagrangian reference must be a parabola (note however that  $u'(1)$  is still unknown). Going back to (5.15) we get the following b.v.p.:

$$\frac{F(e_c)}{e_c} u'(1) \chi u'(\chi) - [F(u(\chi)) u'(\chi)]' = 0, \quad 0 < \chi < 1, \quad (50)$$

$$u(0) = e_r \quad (51)$$

$$u(1) = e_c, \quad (52)$$

An existence and uniqueness theorem is proved in [6] using a shooting technique, i.e. setting  $u'(1) = \alpha$  and replacing (51) by  $u'(1) = \alpha$ . Then the existence of a unique value of  $\alpha$  is proved such that the solution  $u$  of the corresponding Cauchy problem attains the desired value for  $\chi = 0$ .

## 6 Filtration through porous media with hydrophile granules

Babies diapers are porous media, with a cellulose porous matrix, whose pores contain granules which can absorb a volume of water over sixty times their initial volume.

It is essential for good working conditions of the system that the fully swollen grains leave enough space for the liquid to flow. As we shall see, this physical condition plays indeed an important role in showing the solvability of the mathematical model we are going to introduce.

The  $n$ -dimensional model has been described in [9] within a pure Darcyan setting (an approach differing substantially from previous studies [28], [22]).

Let us just summarize the one-dimensional model studied in [8]. The injection rate is given as a function of time at the surface  $x = 0$ . The medium is initially dry with a porosity  $\varepsilon_0$  corresponding to the minimum size of the absorbing granules. The penetration process has two stages: first a non-saturated zone is created and after some (estimable) time  $t_S$  a saturated region appears. For  $t > t_S$  we have two unknowns moving fronts: the wetting front  $x = s(t)$  and the saturation front  $x = \sigma(t)$ .

In [8] only the first stage is considered and an existence theorem is proved for following problem: find the quantities

- $\varepsilon(x, t)$ , porosity,
  - $V(x, t)$ , volume fraction occupied by the granules,
  - $p(x, t)$ , pressure,
  - $s(t)$ , the wetting front, increasing,
- satisfying

$$\frac{\partial \varepsilon}{\partial t} + \frac{\partial V}{\partial t} = 0, \quad (53)$$

$$\frac{\partial V}{\partial t} = f(V_{\max} - V)(S - S_0)_+, \quad (54)$$

$$\frac{\partial(S\varepsilon)}{\partial t} + \frac{\partial q}{\partial x} = -\frac{\partial V}{\partial t} = \frac{\partial \varepsilon}{\partial t} \quad (55)$$

$$S = S(p) \quad (56)$$

$$q(x, t) = -k(S, \varepsilon) \frac{\partial p}{\partial x}, \quad (57)$$

$$-k(S, \varepsilon) \frac{\partial p}{\partial x} \Big|_{x=0} = q_0(t), \quad (58)$$

$$\dot{s}(t) = -\frac{k_0}{S_0 \varepsilon_0} \frac{\partial p}{\partial x} \quad s(0) = 0 \quad (59)$$

$$p(s(t), t) = 0 \quad (S(s(t), t) = S_0), \quad (60)$$

$$\varepsilon(s(t), t) = \varepsilon_0, \quad V(s(t), t) = V_0 \quad (61)$$

with the obvious regularity required for a classical solution.

Equation (53) expresses the conservation of total volume i.e.

$$\varepsilon + V = \varepsilon_0 + V_0. \quad (62)$$

Equation (54) describes the absorption kinetics and  $f$  and is a  $C^1$  increasing function  $f(0) = 0$ , while  $(\cdot)_+$  denotes the positive part. Equation (55) is the fluid mass balance, where the saturation  $S$  is a fraction of  $p$ : for instance one can take

$$S = S_0 + (1 - S_0) \frac{p}{p_S} = S_0 + cp, \quad (63)$$

$S_0$  being the irreducible moisture content. Equation (57) is Darcy's law where  $k$  is  $C^1$ , bounded and strictly positive. In equation (58)  $q_0(t)$  is the inflow rate, it is positive and belongs to  $C^1$ .

The free boundary conditions (59), (60) are not the only possible choice. Here we assume that  $p = 0$  (and therefore  $S = S_0$ ) on the wetting front. However it could make sense to let  $S(s(t), t)$  depend on the speed  $\dot{s}$  of the wetting front, ranging from  $S_0$  to 1. Thus  $S(s(t), t) = S_0$  is a limit case, but it can be replaced by a condition that is known as kinetic condition in the literature about phase change. The other limit case is  $S(s(t), t) = 1$ , making the unsaturated zone disappear.

In order to state the existence theorem for (53)-(61), we first have to perform some manipulation of the equations.

Defining

$$\phi(V) = \int_{V_0}^V \frac{dy}{f(V_{\max} - y)}, \quad V_0 < V < V_{\max} \quad (64)$$

and the inverse function  $\psi$  of  $\phi$ , we integrate (54) and using (62) we obtain the following expression for the porosity

$$\varepsilon(x, t) - \varepsilon_0 = V_0 - \psi \left( \int_{\theta(x)}^t (S(x, \tau) - S_0) d\tau \right), \quad (65)$$

$\theta(x)$  being the inverse function of  $s(t)$ .

From (55), (57), (63) we get the following equation for the pressure

$$\varepsilon c \frac{\partial p}{\partial t} - (1 - S_0 - cp)_+ \frac{\partial \varepsilon}{\partial t} - \frac{\partial}{\partial x} \left[ k \frac{\partial p}{\partial x} \right] = 0 \quad (66)$$

and the problem is reduced to solving (66) with what survives of the boundary conditions (58)-(61)

**Theorem 6.1** *The problem stated above possesses a classical solution global in time, provided that, in addition to the boundeness and regularity assumptions already stated for  $f, q_0, k$ , we suppose that*

$$\frac{\partial k}{\partial S} = 0, \quad \frac{\partial k}{\partial \varepsilon} \geq 0, \quad \frac{\varepsilon}{k} \frac{\partial k}{\partial \varepsilon} < 2, \quad (67)$$

and the function  $f$  is that

$$\frac{1}{\varepsilon_m} \psi'^2 + \psi'' \leq 0, \quad (68)$$

where  $\varepsilon_m = \varepsilon_0 + V_0 - V_{\max}$  is the minimum porosity.

The proof is quite complicated and is fully explained in [8]. Here we remark that the condition  $\frac{\partial k}{\partial S} = 0$  is not physically desirable, but has been inserted for technical reasons (namely in order to estimate  $\frac{\partial p}{\partial t}$ ). The last condition in (67) comes also from technical reasons, but is satisfied by functions  $k(\varepsilon)$  given in the experimental literature (see [4], [3]). The last condition (68) is applied repeatedly in the proof and looks quite essential. If one takes  $f(\xi) = \lambda \xi^n$  ( $\lambda$  and  $n$  positive real numbers), then it simply reduces to

$$\frac{\varepsilon_0}{\varepsilon_m} \geq n + 1, \quad (69)$$

thus setting a lower bound for the minimum porosity, which is a physically meaningful requirement.

The obvious reason of the difficulty of the existence proof is that  $\varepsilon$  is a functional of the pair  $(s, p)$  and appears in critical coefficients. Even proving the seemingly obvious fact that the flux  $-k \frac{\partial p}{\partial x}$  has a constant sign is not trivial.

Uniqueness is still an open question.

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