Optimal Control in Heterogeneous Domain Decomposition Methods for Advection-Diffusion Equations*

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Abstract

New domain decomposition methods (DDM) based on optimal control approach are introduced for the coupling of first and second order equations on overlapping subdomains. Several cost functionals and control functions are proposed. Uniqueness and existence results are proved for the coupled problem, and the convergence of iterative processes is analyzed.

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1 Introduction

In the past decade, there has been a considerable attention to the so-called heterogeneous domain decomposition approach for advection-diffusion equations that are dominated by the advection term: the computational domain is split into two parts, in the one embodying the regions where steep layers occur the original equation is solved in its integrity, whereas in the other the viscous (diffusion) term is dropped, so the problem reduces to its advective part. Obviously, the new heterogeneous differential problem can only be regarded as an approximation of the original one. However, when the two subdomains are separated by a sharp interface, if suitable interface conditions are imposed at the interface itself, then the solution of the reduced heterogeneous problem converges to the one of the original complete problem when the Péclet number (i.e. the ratio between the viscous and the convective term) tends to zero [4]. In this paper we analyse a mathematical formulation of the heterogeneous advection diffusion problem on overlapping subdomains based on an optimal control approach. The optimal control for domain decomposition methods have already been advocated to solve the coupling between heterogeneous equations (see [3, 18, 19]) and they have been analysed to solve homogeneous elliptic problems (see [10, 11, 12]).
The idea consists of introducing a control function on the subdomain interfaces which have the role of guaranteeing that the two solutions match on the region of overlap. The use of control approach for heterogeneous advection-diffusion operators was introduced in [5] for both overlapping and non-overlapping subdomain decompositions. In this paper we generalize the results of [5] in the case of overlapping partitions. From one hand, we introduce a further distributed control function whose support is in the overlapping region. Moreover, we propose iterative methods for the solution of the control problems and analyse their convergence properties.
This paper provides a mathematical set up for the treatment of heterogeneous operators in overlapping subdomains. Some preliminary results concerned with the theory here developed appeared in [2].
Since our analysis is carried out at the differential level, the proposed approach is prone to be adopted in the framework of any kind of numerical approximations (in particular, those based on the finite element method).
An outline of the paper is as follows. In Sect. 2 we introduce the heterogeneous advection-diffusion problem through control at the boundary and we provide conditions on the data and the subdomain decomposition that guarantee the exact controllability (yielding the solutions in the two subdomains that coincide on the overlapping region). In Sect. 3 we formulate the problem as an optimal control problem in which we aim at minimizing the L^2-norm of the error on the overlapping region. We analyse the existence of a solution, then we propose an iterative method and analyse its convergence. In Sect. 4 we introduce a further control function, distributed in the overlapping region. This is a novel approach: we also propose and analyse an iterative method for the approximation of the

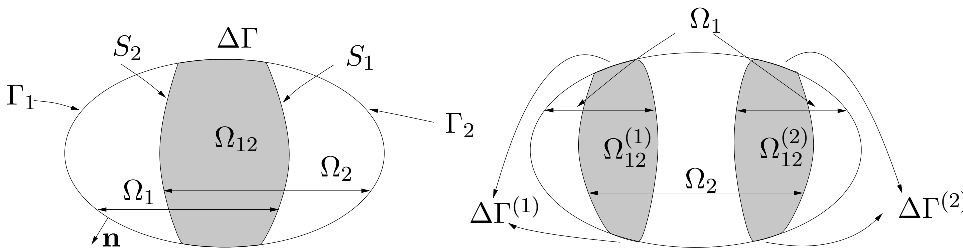

Figure 1: A first possible decomposition, $p = 1$ on the left and $p = 2$ on the right

corresponding solution. In Sect. 5 we compare numerical results obtained with both 2-controls and 3-controls approaches. In Sect. 6 we change the role of the boundary control at the interface: the controls are used to enforce on one interface the continuity of the solution, on the other the continuity of the flux. Finally, in Sect. 7 we apply the previous approach to the case of second order elliptic equations.

2 Problem statements

Let Ω be a two-dimensional domain with the boundary $\Gamma := \partial\Omega$ which is assumed to be Lipschitz-continuous and piecewise of class $C^{(2)}$. Its closure is $\overline{\Omega} = \Omega \cup \Gamma$. We use the following notations (see Fig.1-3 for some examples): Ω_1 and Ω_2 are two, not necessarily connected, subsets of Ω such that

$$\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2, \quad \Omega_1 \cap \Omega_2 \neq \emptyset, \qquad \Omega_{12} = \Omega_1 \cap \Omega_2,$$
$$\Gamma_k = \partial\Omega_k \cap \Gamma, \quad S_k = \partial\Omega_k \setminus \Gamma_k, \quad k = 1, 2.$$

Note that Ω_{12} can be regarded as the union of p disconnected subdomains $\Omega_{12}^{(j)}$ with $j = 1, ..., p$: $\Omega_{12} = \cup_{j=1}^{p} \Omega_{12}^{(j)}$ (see Fig. 1). We consider two situations: when $\Gamma_1 \cap \Gamma_2 \neq \emptyset$ (as in the case of Fig. 1) and $\Gamma_1 \cap \Gamma_2 = \emptyset$ (as in Figs. 2-3). We assume that $\partial\Omega_1, \partial\Omega_2$ are piecewise of class $C^{(2)}$ and Lipschitz-continuous.

Let $\mathbf{n} = (n_1, n_2)$ be the outward unit normal on Γ, $\boldsymbol{\tau} = (n_2, -n_1)$ the tangent vector; $\mathbf{b} = (b_1, b_2)$ is a vector with smooth components. We define

$$b_n^k := \mathbf{b} \cdot \mathbf{n} = \sum_{i=1}^{2} b_i n_i \;\text{ on }\; \partial\Omega_k, \quad b_n^k = (b_n^k)^+ - (b_n^k)^-,$$
$$(b_n^k)^+ = (|b_n^k| + b_n^k)/2, \quad (b_n^k)^- = (|b_n^k| - b_n^k)/2, \quad k = 1, 2,$$

and

$$S_k^- = S_k \cap \{(b_n^k)^- \neq 0\}, \quad S_k^+ = S_k \cap \{(b_n^k)^+ \neq 0\},$$
$$\Gamma_k^- = \Gamma_k \cap \{(b_n^k)^- \neq 0\}, \quad \Gamma_k^+ = \Gamma_k \cap \{(b_n^k)^+ \neq 0\}.$$

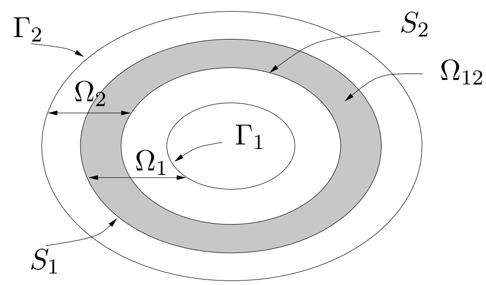

Figure 2: A second possible decomposition with $p = 1$

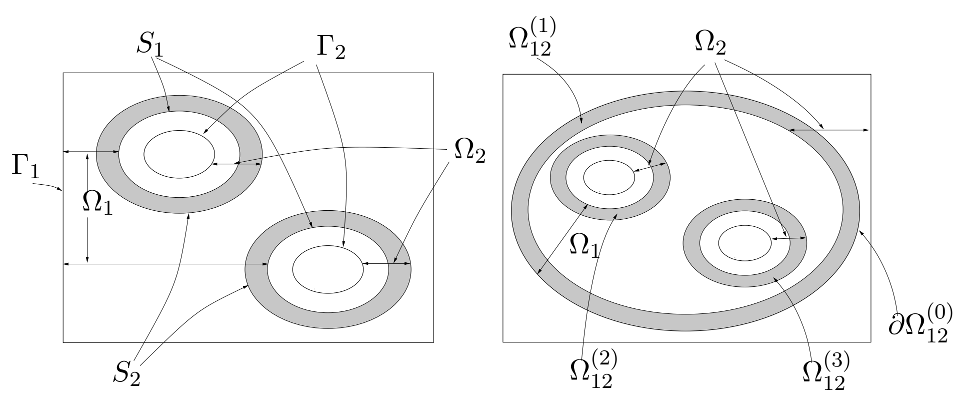

Figure 3: A third possible decomposition with $p = 1$ on the left and $p = 3$ on the right.

We will use the real spaces $L^2(\Omega), L^2(\Omega_k), L^2(\Gamma), \dots, L^2(\Gamma_k),\ k=1,2$ as well as the following spaces: $L^2(w; S_k^-)$ is the space of functions u such that for $k=1,2$

$$L^2(w; S_k^-) \text{ is the space of functions } u: \ \|u\|_{L^2(w;S_k^-)} := \left(\int\limits_{S_k} w|u|^2 d\Gamma \right)^{1/2} < \infty,$$

for $w = (b_n^k)^-$ and $w = (b_n^k)^+$. For simplicity, in the sequel we will use the following notations:

$$L^2(S_k^-) := L^2((b_n^k)^-; S_k^-) \qquad \text{and} \qquad L^2(S_k^+) := L^2((b_n^k)^+; S_k^+).$$

Let us consider the differential operators

$$\begin{aligned} L_1 u_1 :=& \operatorname{div}(\mathbf{b} u_1) + b_0 u_1 \ \text{ in } \ \Omega_1, \\ L_2 u_2 :=& -\nu \Delta u_2 + \operatorname{div}(\mathbf{b} u_2) + b_0 u_2 \ \text{ in } \ \Omega_2, \end{aligned} \tag{1}$$

where $\nu = const > 0$, $\mathbf{b}$ and b_0 are given such that $(b_0 + (\operatorname{div}\mathbf{b})/2) \geq \mu_0 = const > 0,\ \forall x \in \overline{\Omega}$. f is a given function defined in $\overline{\Omega}$, g is a given function defined on $\partial\Omega$ and χ_{12} is the characteristic function of Ω_{12}. In the sequel, the product of a function $u \in L^2(\Omega_{12})$ by χ_{12} will be considered as the prolongation by zero of u onto $\Omega_k \backslash \Omega_{12}$ for $k=1,2$. Moreover we assume that all data $\mathbf{b},\ f_0,\ f,\ g$ in (1) are smooth in $\overline{\Omega}$. Each operator L_k is defined on smooth functions in Ω_k $(k=1,2)$.

We look for the solutions to the problem

$$\begin{aligned} &L_1 u_1 = f \ \text{ in } \ \Omega_1,\ (b_n^1)^- u_1 = (b_n^1)^- g \ \text{ on } \ \Gamma_1, \quad (b_n^1)^- u_1 = (b_n^1)^- \lambda_1 \ \text{ on } \ S_1, \\ &L_2 u_2 = f \ \text{ in } \ \Omega_2,\ u_2 = g \ \text{ on } \ \Gamma_2, \qquad\qquad\qquad u_2 = \lambda_2 \ \text{ on } \ S_2, \end{aligned} \tag{2}$$

where λ_1 and λ_2 are the *controls* and they are chosen so that u_1 and u_2 *adjust* in the best possible way on Ω_{12}, thus we search for *convenient*, both mathematically and physically, conditions on u_1 and u_2 on Ω_{12}.
The *reasonable* request that

$$u_1 = u_2 \qquad \text{in } \Omega_{12} \tag{3}$$

is not necessarily the right answer to our question. As a matter of fact, due to the heterogeneous nature of the problem, the ambition to have $u_1 = u_2$ in Ω_{12} is too strong. For this reason we look for a relaxed form of the condition (3), e.g. in a *least square sense*, that means to minimize the difference $u_1 - u_2$ on Ω_{12}:

$$\inf_{\lambda_1, \lambda_2} J_0(\lambda_1, \lambda_2) \tag{4}$$

where

$$J_0(\lambda_1, \lambda_2) := \frac{1}{2} \int\limits_{\Omega} \chi_{12}(u_1(\lambda_1) - u_2(\lambda_2))^2 d\Omega,$$

and χ_{12} is the characteristic function of Ω_{12}.
Note that in some places we will use also the notation $J_0(u_1, u_2)$ instead of $J_0(\lambda_1, \lambda_2)$, referring to the same cost functional.
Problem (2),(3) is an *exact controllability problem*, while problem (2),(4) is an *optimal control problem*, which can be considered also as a weak statement of (2),(3). We denote by $\boldsymbol{\lambda} = (\lambda_1,\ \lambda_2)$ the vector of "controls" $\lambda_1,\ \lambda_2$.

The minimum problem (4) could be replaced by

$$\inf_{\lambda_1,\lambda_2} J_\alpha(\lambda_1, \lambda_2) = \inf_{\lambda_1,\lambda_2} \left[\frac{1}{2} \left(\alpha \int\limits_{S_1} (b_n^1)^- \lambda_1^2 d\Gamma + \alpha \int\limits_{S_2} \lambda_2^2 d\Gamma \right) + J_0(\lambda_1, \lambda_2) \right], \quad \alpha \geq 0. \tag{5}$$

In this case, we name (2),(5) the *regularized* version of (2),(4).

Statements of both problems (2),(3) and (2),(4) use the decomposition of Ω onto Ω_1, Ω_2 with overlapping and will therefore be considered as heterogeneous domain decomposition methods (DDM) with overlapping.

3 Analysis of DDM with two control functions

The variational equations ("optimality conditions") corresponding to problem (2),(4) read as follows: find u_1, u_2, λ_1, λ_2, q_1 and q_2 such that

$$\begin{cases} L_1 u_1 = f \ \text{ in } \ \Omega_1, \\ \qquad (b_n^1)^- u_1 = (b_n^1)^- g \ \text{ on } \ \Gamma_1, \quad (b_n^1)^- u_1 = (b_n^1)^- \lambda_1 \ \text{ on } \ S_1, \\ L_2 u_2 = f \ \text{ in } \ \Omega_2, \quad u_2 = g \ \text{ on } \ \Gamma_2, \quad u_2 = \lambda_2 \ \text{ on } \ S_2, \\ L_1^{(0)*} q_1 = \chi_{12}(u_1 - u_2) \ \text{ in } \ \Omega_1, \\ \qquad (b_n^1)^+ q_1 = 0 \ \text{ on } \ \Gamma_1, \quad (b_n^1)^+ q_1 = 0 \ \text{ on } \ S_1, \\ L_2^{(0)*} q_2 = -\chi_{12}(u_1 - u_2) \ \text{ in } \ \Omega_2, \quad q_2 = 0 \ \text{ on } \ \partial\Omega_2, \\ (b_n^1)^- q_1 = 0 \ \text{ on } \ S_1, \quad -\nu \dfrac{\partial q_2}{\partial n} = 0 \ \text{ on } \ S_2, \end{cases} \tag{6}$$

where the operator $L_k^{(0)}$ (for $k = 1, 2$) is defined as in (1) on smooth functions which satisfy homogeneous boundary conditions on $\partial\Omega_k$, while $L_k^{(0)*}$ is its adjoint operator.

We consider problem (2) and suppose that λ_k (for $k = 1, 2$) are known. For $k = 1, 2$ we decompose the solutions u_k in a linear plus an affine part $u_k = u_k^\lambda + u_k^f$

as follows:

$$
\begin{aligned}
&L_1 u_1^f = f \ \text{ in } \ \Omega_1,\\
&\qquad (b_n^1)^- u_1^f = (b_n^1)^- g \ \text{ on } \ \Gamma_1, \quad (b_n^1)^- u_1^f = (b_n^1)^- \tilde{g}_1 \ \text{ on } \ S_1,\\
&L_2 u_2^f = f \ \text{ in } \ \Omega_2, \quad u_2^f = g \ \text{ on } \ \Gamma_2, \quad u_2^f = \tilde{g}_2 \ \text{ on } \ S_2,\\
&L_1 u_1^\lambda = 0 \ \text{ in } \ \Omega_1,\\
&\qquad (b_n^1)^- u_1^\lambda = 0 \ \text{ on } \ \Gamma_1, \quad (b_n^1)^- u_1^\lambda = (b_n^1)^- \tilde{\lambda}_1 \ \text{ on } \ S_1,\\
&L_2 u_2^\lambda = 0 \ \text{ in } \ \Omega_2, \quad u_2^\lambda = 0 \ \text{ on } \ \Gamma_2, \quad u_2^\lambda = \tilde{\lambda}_2 \ \text{ on } \ S_2,
\end{aligned}
\tag{7}
$$

where $\tilde{g}_k$ is a given extension of g onto S_k (for $k = 1, 2$), such that

$$
\begin{aligned}
&g_2 := \{g \text{ on } \Gamma_2, \ \tilde{g}_2 \text{ on } S_2\} \in H^{1/2}(\partial\Omega_2),\\
&g_1 := \{g \text{ on } \Gamma_1, \ \tilde{g}_1 \text{ on } S_1\} \in L^2(S_1^- \cup \Gamma_1^-),\\
&\tilde{\lambda}_k := \lambda_k - \tilde{g}_k \text{ on } S_k, \quad u_k^\lambda := u_k(\tilde{\lambda}_k), \quad k = 1, 2.
\end{aligned}
\tag{8}
$$

Now (5) has the following formi for any $\alpha \geq 0$:

$$
\inf_{\tilde{\lambda}_1, \tilde{\lambda}_2} \left[\frac{1}{2} \left(\alpha \int_{S_1} (b_n^1)^- (\tilde{\lambda}_1 + \tilde{g}_1)^2 d\Gamma + \alpha \int_{S_2} (\tilde{\lambda}_2 + \tilde{g}_2)^2 d\Gamma \right) + J_0(\tilde{\lambda}_1, \tilde{\lambda}_2) \right], \tag{9}
$$

where

$$
J_0(\tilde{\lambda}_1, \tilde{\lambda}_2) := \frac{1}{2} \int_\Omega \chi_{12} (u_1(\tilde{\lambda}_1) - u_2(\tilde{\lambda}_2) - F)^2 d\Omega, \quad F = -\chi_{12}(u_1^f - u_2^f). \tag{10}
$$

Setting $D(A) := \Lambda = L^2(S_1^-) \times H_{00}^{1/2}(S_2)$, we define the linear operator

$$
A : L^2(S_1^-) \times L^2(S_2) \to L^2(\Omega_{12}), \quad A\tilde{\boldsymbol{\lambda}} := \chi_{12}(u_1^\lambda - u_2^\lambda), \tag{11}
$$

and its adjoint operator

$$
A^* : L^2(\Omega_{12}) \to L^2(S_1^-) \times L^2(S_2), \quad A^* : w \mapsto \boldsymbol{\mu}, \tag{12}
$$

such that:

$$
\begin{aligned}
&L_1^{(0)*} q_1 = \chi_{12} w \ \text{ in } \ \Omega_1, \quad (b_n^1)^+ q_1 = 0 \ \text{ on } \ \partial\Omega_1,\\
&L_2^{(0)*} q_2 = -\chi_{12} w \ \text{ in } \ \Omega_2, \quad q_2 = 0 \ \text{ on } \ \partial\Omega_2,\\
&\boldsymbol{\mu} = (\mu_1, \mu_2), \quad \mu_1 = (b_n^1)^- q_{1|S_1}, \quad \mu_2 = -\nu \frac{\partial q_2}{\partial n}|_{S_2},
\end{aligned}
$$

and

$$
F = -\chi_{12}(u_1^f - u_2^f). \tag{13}
$$

Problem (2),(3) can be written as

$$\text{find } \tilde{\boldsymbol{\lambda}} \in \Lambda : \qquad A\tilde{\boldsymbol{\lambda}} = F, \tag{14}$$

while problem (6) can be written as

$$\text{find } \tilde{\boldsymbol{\lambda}} \in \Lambda : \qquad A^*A\tilde{\boldsymbol{\lambda}} = A^*F. \tag{15}$$

In this section we analyse the well-posedeness of this latter problem. The study of existence and uniqueness of solution for problem (2),(4) will be carried out by using the properties of problem (2),(3).

Let us prove the first proposition.

Proposition 3.1 *Problem (2),(3) does not have a solution in general, without introducing specific restriction on the data of the problem itself.*

Proof. The proof is based on the uniqueness continuation theorem in the case of Figure 1, and on the index theory for cases like those illustrated in Figure 2 and Figure 3.

Assume that Ω_{12} is a connected set, $\Delta\Gamma = \Gamma_1 \cap \Gamma_2$, $\Delta\Gamma \subset \partial\Omega_{12}$, $\text{meas}(\Delta\Gamma) > 0$, $b_n \neq 0$ on $\Delta\Gamma$, $g \equiv 0$ on Γ, $f \equiv 0$ on $\Delta\Gamma$, $f > 0$ in Ω_{12} and (2),(3) has a solution $\{u_1,\ u_2,\ \lambda_1,\ \lambda_2\}$. Note that, if $\{u_1,\ u_2,\ \lambda_1,\ \lambda_2\}$ is a smooth solution of (2),(3) then we can consider the equation $L_1 u_1 = f$ on $\Delta\Gamma$ and find

$$b_n \frac{\partial u_1}{\partial n} + b_\tau \frac{\partial u_1}{\partial \tau} + \mu u_1 = f \text{ on } \Delta\Gamma,$$

where $\mu = b_0 + \text{div}\mathbf{b}$, $\quad b_n = \mathbf{b}\cdot\mathbf{n}$, $\quad b_\tau = \mathbf{b}\cdot\boldsymbol{\tau}$, and $\mathbf{n}$ is outward unit normal to $\Delta\Gamma$. Since $u_1 = u_2 = u$ in Ω_{12} from the equations in Ω_1, Ω_2 we obtain:

$$L_1 u = f \text{ in } \Omega_{12}, \quad u = 0 \text{ on } \Delta\Gamma, \quad \nu\Delta u = 0 \text{ in } \Omega_{12},$$
$$L_1 u|_{\Delta\Gamma} = f|_{\Delta\Gamma} = 0 \Longrightarrow \quad b_n \frac{\partial u}{\partial n}\bigg|_{\Delta\Gamma} + \mu u|_{\Delta\Gamma} + b_\tau \frac{\partial u}{\partial \tau}\bigg|_{\Delta\Gamma} = b_n \frac{\partial u}{\partial n}\bigg|_{\Delta\Gamma} = 0.$$

So, u would be the solution of following Cauchy problem:

$$\Delta u = 0 \text{ in } \Omega_{12}, \quad u = \frac{\partial u}{\partial n} = 0 \text{ on } \Delta\Gamma.$$

According to the uniqueness continuation theorem [7], $u = 0$ in Ω_{12}. Hence, $u_k = 0$ in Ω_{12}, $\lambda_k = 0,\ k = 1, 2$ and $f = 0$ in Ω_{12}, which contradicts the assumption $f > 0$ in Ω_{12}.

Let us now consider another counter-example. Assume that Ω_{12} is a multi connected domain with boundary $\partial\Omega_{12} = \bigcup_{j=0}^{p} \partial\Omega_{12}^{(j)}$, where $\partial\Omega_{12}^{(0)}$ is the outer part of $\partial\Omega_{12}$ (see Figs. 2 - 3, in particular $\partial\Omega_{12}^{(0)} = S_1$ in Fig. 2).

Suppose that $|\mathbf{b}| \neq 0$ on $\partial\Omega_{12}$ and

$$æ = 2(P - p + 1) < 0, \text{ where } P = \frac{1}{2\pi} \sum_{j=0}^{p} \{\arg(b_1 - ib_2)\},$$

where æ is the "index" of the problem considered in Ω_{12} [16]. Assume that problem (2),(3) has a solution, then $u = u_1 = u_2$ is the solution in Ω_{12} of the following Poincaré problem:

$$\Delta u = 0 \text{ in } \Omega_{12}, \qquad \mathbf{b} \cdot \nabla u + \mu u = f \text{ on } \partial\Omega_{12}.$$

Existence of a solution for this problem can be proved if we impose a number of restrictions on f and the "index" æ must be nonnegative. There are further counter-examples which prove the non-existence of solutions of (2),(3) in general case. □

Introduce the following types of assumption:

I.

$$\begin{cases} \Omega_{12} = \bigcup\limits_{j=1}^{p} \Omega_{12}^{(j)} \quad \Delta\Gamma = \bigcup\limits_{j=1}^{p} \Delta\Gamma^{(j)}, \quad \Delta\Gamma^{(j)} \subset \partial\Omega_{12}^{(j)}, \quad p < \infty, \\ \text{meas}(\Delta\Gamma^{(j)}) > 0, \ b_n \neq 0 \text{ on } \Delta\Gamma^{(j)}, \ (\text{see Fig. 1, with } p = 1 \text{ or } p = 2), \end{cases} \tag{16}$$

II.

$$\begin{cases} \Omega_{12} \text{ is finite, } \mu = b_0 + \text{div}\mathbf{b} \geq 0 \text{ on } \partial\Omega_{12}, \ \mu \not\equiv 0 \text{ on } \partial\Omega_{12}, \\ \text{the direction } \mathbf{b} \text{ at any point of } \partial\Omega_{12} \text{ forms with the outward normal} \\ \text{to } \partial\Omega_{12} \text{ an acute angle.} \end{cases} \tag{17}$$

III.

$$\begin{cases} \Omega_{12} = \bigcup\limits_{j=1}^{p} \Omega_{12}^{(j)}, \ b_n \neq 0 \text{ on } \partial\Omega_{12}, \ \dfrac{\mu}{b_n} - \dfrac{1}{2}\dfrac{\partial}{\partial\tau}\left(\dfrac{b_\tau}{b_n}\right) > 0 \text{ on } \partial\Omega_{12}, \\ \text{where } \dfrac{\partial}{\partial\tau} \text{ is the derivative along } \partial\Omega_{12} \end{cases} \tag{18}$$

Let us prove the second proposition.

Proposition 3.2 *If problem (2),(3) has a solution and one of the assumptions* I-III *is fulfilled, then this solution is unique.*

Proof. Let us assume that the data are such that problem (2),(3) admits at least one solution.

Let the assumption (16) be fulfilled and $\{u_1^{(1)}, u_2^{(1)}, \lambda_1^{(1)}, \lambda_2^{(1)}\}$, $\{u_1^{(2)}, u_2^{(2)}, \lambda_1^{(2)}, \lambda_2^{(2)}\}$ be two possible solutions of (2),(3). By setting $u_1 = u_1^{(1)} - u_1^{(2)}$, $u_2 = u_2^{(1)} - u_2^{(2)}$, $\lambda_1 = \lambda_1^{(1)} - \lambda_1^{(2)}$ and $\lambda_2 = \lambda_2^{(1)} - \lambda_2^{(2)}$, we obtain the following boundary value problems in Ω_{12} for $u = u_1 = u_2$:

$$\Delta u = 0 \text{ in } \Omega_{12}, \quad b_n \frac{\partial u}{\partial n} + b_\tau \frac{\partial u}{\partial \tau} + \mu u = 0 \text{ on } \partial\Omega_{12}, \tag{19}$$

$$\Delta u = 0 \text{ in } \Omega_{12}^{(j)}, \quad u = \frac{\partial u}{\partial n} = 0 \text{ on } \Delta\Gamma^{(j)}, \ j = 1, \ldots, p. \tag{20}$$

From (20) and the uniqueness continuation theorem we obtain: $u = 0$ in $\Omega_{12}^{(j)}$, $j = 1, \ldots, p$. Hence, $\lambda_1 = 0, \lambda_2 = 0$ and $u_1 = 0$ in Ω_1, $u_2 = 0$ in Ω_2, i.e. the solution of (2),(3) is unique.

Suppose now that assumption (17) is fulfilled. From the theory of boundary-value problems with oblique derivative [15], we conclude that problem (19) has a trivial solution. Then: $\lambda_k = 0$, $u_k = 0$ in Ω_k, $k = 1, 2$ and the uniqueness of solution (2),(3) takes place.

Finally, let the assumption (18) be valid. Then for the solution of (19) we have the following well-known relations:

$$0 = -\int_{\Omega_{12}} \Delta u u d\Omega = \int_{\Omega_{12}} |\nabla u|^2 d\Omega - \int_{\partial\Omega_{12}} \frac{\partial u}{\partial n} u d\Gamma.$$

Using boundary condition from (19) we obtain:

$$0 = \int_{\Omega_{12}} |\nabla u|^2 d\Omega + \int_{\partial\Omega_{12}} \left(\frac{b_\tau}{b_n} \frac{\partial u}{\partial \tau} + \frac{\mu}{b_n} u \right) u d\Gamma =$$
$$= \int_{\Omega_{12}} |\nabla u|^2 d\Omega + \int_{\partial\Omega_{12}} \left(\frac{\mu}{b_n} - \frac{1}{2} \frac{\partial}{\partial \tau} \left(\frac{b_\tau}{b_n} \right) \right) u^2 d\Gamma.$$

Hence, $u = 0$ in Ω_{12}, $\lambda_k = 0$, $u_k = 0$ in Ω_k, $k = 1, 2$ and the solution of (2),(3) is unique. □

Remark 3.1 *Since, for a linear operator A, $Ker(A) = Ker(A^*A)$, then the statement of Prop. 3.2 gives also the uniqueness of solution of problem (2),(4) or equivalently of (6).*

The assertions of Propositions 3.1 - 3.2 will be used in the next Proposition while analysing problem (2),(4). Let us note also that analogous assertions can be proved for the case $\Omega \subset \mathbb{R}^n$, $n > 2$, and for a system of equations of type (2),(3). The results of the Cauchy problems, the problems of the oblique derivative and Poincare problem are still useful in proving these assertions.

Proposition 3.3 *The following assertions hold true.*

1. *Problem (2),(4) (or equivalently (6)) has not a solution in general.*
2. *If problem (2),(4) (or equivalently (6)) has a solution, then in general case $\inf_\lambda J_0(\lambda_1, \lambda_2) > 0$, i.e. $u_1 \neq u_2$ in Ω_{12}.*
3. *If one of the assumptions* I-III *is satisfied and problem (2),(4) (or equivalently (6)) has a solution then this solution is unique.*

Proof. *1.* Consider the following adjoint problem with homogeneous boundary conditions: find q_1, q_2, w s.t.

$$\left\{ \begin{array}{l} L_1^{(0)*} q_1 = \chi_{12} w \text{ in } \Omega_1, \quad (b_n^1)^+ q_1 = 0 \text{ on } \partial\Omega_1, \\ L_2^{(0)*} q_2 = -\chi_{12} w \text{ in } \Omega_2, \quad q_2 = 0 \text{ on } \partial\Omega_2, \\ (b_n^1)^- q_1 = 0 \text{ on } S_1, \quad -\nu \dfrac{\partial q_2}{\partial n} = 0 \text{ on } S_2. \end{array} \right. \tag{21}$$

To simplify notations let us replace $-q_2$ by q_2 in (21).
Assume that $\text{meas}(S_1^-) = 0$, in this case the control function λ_1 is not needed in the solution of problem (2),(4). Let us consider any smooth function $\tilde{q}_2$ defined on Ω_2 with compact support in Ω_{12} such that $\text{dist}(\partial\Omega_{12}, \text{supp}(\tilde{q}_2)) \geq \varepsilon = \textit{const} > 0$, where $0 < \varepsilon \ll \dim(\Omega_{12})$.
Let $\tilde{w}$ be defined on Ω_{12} such that $\tilde{w} = \chi_{12} L_2^{(0)*} \tilde{q}_2$. Note that $\text{supp}(\tilde{w}) \subset \Omega_{12}$, then we extend it by zero in Ω_1 and solve $L_1^{(0)*} \tilde{q}_1 = \chi_{12} \tilde{w}$.
It is easy to see, that $\tilde{q}_1 \neq \tilde{q}_2$ in Ω_{12}, in general.
We anticipate that the relation "$(b_n^1)^- \tilde{q}_1 = 0$ on S_1" does not provide any information since $(b_n^{-1})^- = 0$ on S_1, so the adjoint problem (21) has an infinite set of nontrivial solutions $\tilde{q}_1,\ \tilde{q}_2,\ \tilde{w}$.
Now, suppose that $g \equiv 0$ in (2),(3) and let us consider f as the solution of the following equation: $Tf := \chi_{12}((L_1^{(0)})^{-1} f - (L_2^{(0)})^{-1} f) = \tilde{w}$ in Ω. It is easy to see that $\ker(T) = \{0\}$.
Assume now that, for this f, problem (2),(4) has a solution $u_k = u_k^\lambda + u_k^f$, for $k = 1, 2$, where u_k^λ is the linear component of u_k generated by $\tilde{\lambda}_k$, while u_k^f is generated by both f and g_k. Then

$$J_0(\lambda_1, \lambda_2) = \frac{1}{2} \|\chi_{12}(u_1(\lambda_1) - u_2(\lambda_2))\|^2_{L^2(\Omega)} = \frac{1}{2} \|A\tilde{\boldsymbol{\lambda}} - F\|^2_{L^2(\Omega)},$$

Since $L_k^{(0)} u_k(\tilde{\lambda}_k) = 0$, for $k = 1, 2$, we have

$$\begin{aligned} (A\tilde{\boldsymbol{\lambda}}, F)_{L^2(\Omega_{12})} &= (\chi_{12} u_1(\tilde{\lambda}_1), \tilde{w})_{L^2(\Omega)} - (\chi_{12} u_2(\tilde{\lambda}_2), \tilde{w})_{L^2(\Omega)} \\ &= (u_1(\tilde{\lambda}_1), L_1^{(0)*} \tilde{q}_1)_{L^2(\Omega_1)} - (u_2(\tilde{\lambda}_2), L_2^{(0)*} \tilde{q}_2)_{L^2(\Omega_2)} = 0. \end{aligned} \tag{22}$$

Therefore $F \in \ker(A^*)$ (remember that $L_2(\Omega_{12}) = \overline{R(A)} \oplus \ker(A^*)$), the functional J_0 takes the following form

$$J_0(\lambda_1, \lambda_2) = \frac{1}{2} \|\chi_{12}(u_1(\lambda_1) - u_2(\lambda_2))\|^2_{L^2(\Omega)} = \frac{1}{2} \left(\|A\tilde{\boldsymbol{\lambda}}\|^2_{L^2(\Omega)} + \|F\|^2_{L^2(\Omega)} \right)$$

and problem (2),(4) has not a solution either [21]. In other words, we note that $\tilde{\boldsymbol{\lambda}}$ is the solution of $\inf_{\tilde{\lambda}_1, \tilde{\lambda}_2} J_0(\tilde{\lambda}_1, \tilde{\lambda}_2)$ iff there exists $\tilde{\boldsymbol{\lambda}}$ such that $A^* A \tilde{\boldsymbol{\lambda}} = A^* F$. But (22) implies that $F := \tilde{w} \in R(A)^\perp = \overline{R(A)}^\perp$ [6, pag. 58] and then $\overline{R(A)}^\perp \neq \{0\}$. Recalling that $L^2(\Omega_{12}) = \overline{R(A)} \oplus \overline{R(A)}^\perp$, it follows that $F \notin \overline{R(A)}$, thus (6) (and therefore (2),(4)) might not have a solution.

2. Assume that problem (2),(4) has a solution such that $J_0(\lambda_1, \lambda_2) = 0$. Then $u_1 = u_2$ in Ω_{12}. However we know from Prop. 2.1 that this is not possible in the general case.

3. We note that (2),(4) is equivalent to (6) and that (6) has a unique solution if and only if (2),(3) has a unique solution; then, from Prop. 2.2, if (2),(4) has a solution and I-III are satisfied, then this solution is unique.

□

3.1 Iterative algorithm

The simpler way to solve (6) (or equivalently (15)) is to consider the gradient method applied to the minimization problem $\inf\limits_{\tilde{\lambda}_1, \tilde{\lambda}_2} J_0(\tilde{\lambda}_1, \tilde{\lambda}_2)$. It reads: for a given $\tilde{\boldsymbol{\lambda}}^0$, for $m = 0, 1, ..., $ find $\tilde{u}_k^m$, $\tilde{\lambda}_k^{m+1}$ $(k = 1, 2)$, such that

$$\begin{cases} L_1\tilde{u}_1^m = 0 \;\text{ in }\; \Omega_1,\; (b_n^1)^-\tilde{u}_1^m = 0 \;\text{ on }\; \Gamma_1,\; (b_n^1)^-\tilde{u}_1^m = (b_n^1)^-\tilde{\lambda}_1^m \;\text{ on }\; S_1, \\ L_2\tilde{u}_2^m = 0 \;\text{ in }\; \Omega_2,\; \tilde{u}_2^m = g \;\text{ on }\; \Gamma_2,\; \tilde{u}_2^m = \tilde{\lambda}_2^m \;\text{ on }\; S_2, \\ \tilde{\boldsymbol{\lambda}}^{m+1} = \tilde{\boldsymbol{\lambda}}^m - \gamma_m J_0'(\tilde{\lambda}_1^m, \tilde{\lambda}_2^m), \end{cases} \tag{23}$$

where $\{\gamma_m\}$ are suitable relaxation parameters to be chosen according to convergence criteria ([1, 14, 17, 20, 22]).
For the functions $u_k = u_k^\lambda + u_k^f$, with $\lambda_k = \tilde{\lambda}_k + \tilde{g}_k$, $(k = 1, 2)$ the gradient method reads: for a given $\boldsymbol{\lambda}^0$, for $m = 0, 1, ...,$ find u_1^m, u_2^m, λ_1^{m+1} and λ_2^{m+1} solutions of

$$\begin{cases} L_1u_1^m = f \;\text{ in }\; \Omega_1, \\ \quad (b_n^1)^-u_1^m = (b_n^1)^-g \;\text{ on }\; \Gamma_1, \qquad (b_n^1)^-u_1^m = (b_n^1)^-\lambda_1^m \;\text{ on }\; S_1, \\ L_2u_2^m = f \;\text{ in }\; \Omega_2, \quad u_2^m = g \;\text{ on }\; \Gamma_2, \quad u_2^m = \lambda_2^m \;\text{ on }\; S_2, \\ \boldsymbol{\lambda}^{m+1} = \boldsymbol{\lambda}^m - \gamma_m J_0'(u_1^m, u_2^m). \end{cases} \tag{24}$$

If the relaxation parameters $\{\gamma_m\}$ are chosen to satisfy a minimization procedure, algorithm (23) is in fact a minimization procedure for solving (2),(4) and it can be rewritten as: for a given $\boldsymbol{\lambda}^0$, for $m = 0, 1, ...,$ find u_1^m, u_2^m, λ_1^{m+1}, λ_2^{m+1}, q_1^m and q_2^m such that

$$\begin{cases} L_1u_1^m = f \;\text{ in }\; \Omega_1, \\ \qquad (b_n^1)^-u_1^m = (b_n^1)^-g \;\text{ on }\; \Gamma_1, \quad (b_n^1)^-u_1^m = (b_n^1)^-\lambda_1^m \;\text{ on }\; S_1, \\ L_2u_2^m = f \;\text{ in }\; \Omega_2, \quad u_2^m = g \;\text{ on }\; \Gamma_2, \quad u_2^m = \lambda_2^m \;\text{ on }\; S_2, \\ L_1^{(0)*}q_1^m = \chi_{12}(u_1^m - u_2^m) \;\text{ in }\; \Omega_1, \quad (b_n^1)^+q_1^m = 0 \;\text{ on }\; \partial\Omega_1, \\ L_2^{(0)*}q_2^m = -\chi_{12}(u_1^m - u_2^m) \;\text{ in }\; \Omega_2, \quad q_2^m = 0 \;\text{ on }\; \partial\Omega_2, \\ (b_n^1)^-\lambda_1^{m+1} = (b_n^1)^-\lambda_1^m - \gamma_m(b_n^1)^-q_1^m \;\text{ on }\; S_1, \\ \lambda_2^{m+1} = \lambda_2^m + \gamma_m\nu\,\frac{\partial q_2^m}{\partial n} \;\text{ on }\; S_2. \end{cases} \tag{25}$$

According to the general theory of iterative methods [14], if $ker(A^*A) = \{0\}$, the coefficients γ_m can be chosen in the interval $\gamma_m \in (0, 2/\|A\|^2)$. If problem (2),(3) is dense solvable (see Definition 4.1 in Section 4), the choice [1]

$$\gamma_m = \frac{1}{2}\frac{J_0(\lambda_1^m, \lambda_2^m)}{\int_{S_1}(b_n^1)^-(q_1^m)^2 d\Gamma + \int_{S_2}(\nu\frac{\partial q_2^m}{\partial n})^2 d\Gamma} \tag{26}$$

is the one that guarantees the minimization of functional (4).
Alternative strategies are usable as well, for instance, solving system (15) by the Conjugate Gradient method, in which case the choice of the relaxation parameters is automatic.

Proposition 3.4 *If problem (2),(4) (or equivalently (6)) has a unique solution and the iterative process (23) (or equivalently (25)) is convergent, then in general*

$$\lim_{m\to\infty} \|u_1^m - u_2^m\|_{L_2(\Omega_{12})} \geq \text{const} > 0, \tag{27}$$

i.e. u_1^m, u_2^m *don't coincide in* Ω_{12}*, in general, as* $m \to \infty$.

Proof. This statement follows from both second assertion of proposition 3.3 and convergence of the iterative process (24) (or equivalently (25)) when the relaxation parameters γ_m are chosen appropriately [1, 14, 22]. □

For any $\alpha > 0$, the regularized problem (2),(5) has a unique solution $\{u_1(\alpha), u_2(\alpha), \lambda_1(\alpha), \lambda_2(\alpha)\}$. As a matter of fact, the corresponding optimality conditions are given by (6) where the last row is replaced by

$$\alpha(b_n^1)^- \lambda_1 + (b_n^1)^- q_1 = 0 \text{ on } S_1, \quad \alpha\lambda_2 - \nu\frac{\partial q_2}{\partial n} = 0 \text{ on } S_2, \tag{28}$$

and coherently, equation (15) will be replaced by

$$\text{find } \tilde{\boldsymbol{\lambda}} \in \Lambda : \qquad (\alpha I + A^*A)\tilde{\boldsymbol{\lambda}} = A^*F. \tag{29}$$

It is evident that problem (29) is well posed for any $\alpha > 0$.
The associated iterative process will converge: $u_k^m(\alpha) \to u_k(\alpha)$, $\lambda_k^m(\alpha) \to \lambda_k(\alpha)$ when $m \to \infty$, for $k = 1, 2$ and for any $\alpha > 0$ (m denotes the iteration in the iterative process). Nevertheless, we cannot prove that $u_k(\alpha) \to u_k^{(0)}$, $\lambda_k(\alpha) \to \lambda_k^{(0)}$ when $\alpha \to 0$, for $k = 1, 2$, where $\{u_1^{(0)}, u_2^{(0)}, \lambda_1^{(0)}, \lambda_2^{(0)}\}$ is the solution of (2),(4).
As a matter of fact, if problem (2),(4) admits several solutions, then $\{u_1(\alpha), u_2(\alpha), \lambda_1(\alpha), \lambda_2(\alpha)\}$ will converge to that solution that minimizes the norm of (λ_1, λ_2) (see [1]). However, if problem (2),(4) has not a solution, we can not expect the convergence of the iterative process.
From Proposition 3.4 we can draw the following conclusion: in order for the property $\lim\limits_{m\to\infty} \|u_1^m - u_2^m\| = 0$ to hold, the statement (2),(4) has to be modified. One possibility, which consists of introducing a third control (besides λ_1 and λ_2), will be investigated in the next section.

4 DDM with three control functions

In this section we propose and analyze a domain decomposition algorithm to *approximately* solve problem (2),(3) with a perturbed equation in Ω_1, by making use of three control functions.

Let ω be a smooth function in Ω such that

$$0 \leq \omega(x) \leq 1 \text{ in } \Omega, \quad \omega = 0 \text{ in } \Omega\backslash\Omega_{12}, \quad \omega > 0 \text{ in } \Omega_{12}.$$

Let us consider the following control problem: find $u_k,\ \lambda_k$ for $k = 1,2$ and $v \in L_2(\Omega_{12})$ s.t.

$$\begin{aligned} &L_1 u_1 = f + \omega v \ \text{ in } \ \Omega_1, \\ &\qquad (b_n^1)^- u_1 = (b_n^1)^- g \ \text{ on } \ \Gamma_1, \quad (b_n^1)^- u_1 = (b_n^1)^- \lambda_1 \ \text{ on } \ S_1, \\ &L_2 u_2 = f \ \text{ in } \ \Omega_2, \quad u_2 = g \ \text{ on } \ \Gamma_2, \quad u_2 = \lambda_2 \ \text{ on } \ S_2, \end{aligned} \tag{30}$$

with the constraint

$$u_1 = u_2 \ \text{ in } \ \Omega_{12}. \tag{31}$$

The *optimal* control problem that we associate with (30),(31) reads: find $u_k(\alpha)$, $\lambda_k(\alpha)$, for $k = 1,2$ and $v(\alpha)$ (that we will still denote for simplicity as $u_k,\ \lambda_k$ (for $k = 1,2$), and v) which satisfy both boundary value problem (30) and

$$\inf_{\lambda_1,\lambda_2,v} J_\alpha(\lambda_1,\lambda_2,v), \tag{32}$$

where

$$J_\alpha(\lambda_1,\lambda_2,v) = \frac{1}{2}\left(\alpha\int_{S_1}(b_n^1)^-\lambda_1^2 d\Gamma + \alpha\int_{S_2}\lambda_2^2 d\Gamma + \alpha\int_{\Omega}\omega v^2 d\Omega + \int_{\Omega}\chi_{12}(u_1-u_2)^2 d\Omega\right), \tag{33}$$

and $\alpha \geq 0$ is a regularization parameter. (Even for $\alpha = 0$ the solution to (30),(32) does not necessarily coincide with that of (30),(31).)

In the sequel we identify $L^2(\Omega_{12})$ with the subspaces $L_0^2(\Omega_k) = \{u : u \in L^2(\Omega_k),\ u \equiv 0 \text{ in } \Omega_k \backslash \Omega_{12}\}$, for both $k = 1,2$.

If $\alpha = 0$, (30),(32) represent the weak statement of problem (30),(31). The minimization requirement (32) yields the set of *optimality conditions* find $\lambda_1,\ \lambda_2,\ v,\ u_1,\ u_2,\ q_1,\ q_2$ such that

$$\left\{\begin{aligned} &L_1 u_1 = f + \omega v \ \text{ in } \ \Omega_1, \\ &\qquad (b_n^1)^- u_1 = (b_n^1)^- g \ \text{ on } \ \Gamma_1, \quad (b_n^1)^- u_1 = (b_n^1)^- \lambda_1 \ \text{ on } \ S_1, \\ &L_2 u_2 = f \ \text{ in } \ \Omega_2, \quad u_2 = g \ \text{ on } \ \Gamma_2, \quad u_2 = \lambda_2 \ \text{ on } \ S_2, \\ &L_1^{(0)*} q_1 = \chi_{12}(u_1 - u_2) \ \text{ in } \ \Omega_1, \quad (b_n^1)^+ q_1 = 0 \ \text{ on } \ \partial\Omega_1, \\ &L_2^{(0)*} q_2 = -\chi_{12}(u_1 - u_2) \ \text{ in } \ \Omega_2, \quad q_2 = 0 \ \text{ on } \ \partial\Omega_2, \\ &\alpha (b_n^1)^- \lambda_1 + (b_n^1)^- q_1 = 0 \ \text{ on } \ S_1, \\ &\qquad \alpha\lambda_2 - \nu \tfrac{\partial q_2}{\partial n} = 0 \ \text{ on } \ S_2, \quad \alpha\omega v + \omega q_1 = 0 \text{ in } \Omega_1. \end{aligned}\right. \tag{34}$$

We consider the Gradient method to solve (34): for any given $\lambda_1^0,\ \lambda_2^0,\ v^0$, for

$m \geq 0$ we look for u_1^m, u_2^m, q_1^m, q_2^m, λ_1^{m+1}, λ_2^{m+1}, v^{m+1} such that

$$\left\{\begin{array}{l} L_1 u_1^m = f + \omega v^m \quad \text{in} \quad \Omega_1, \\ \qquad (b_n^1)^- u_1^m = (b_n^1)^- g \quad \text{on} \quad \Gamma_1, \quad (b_n^1)^- u_1^m = (b_n^1)^- \lambda_1^m \quad \text{on} \quad S_1, \\ L_2 u_2^m = f \quad \text{in} \quad \Omega_2, \quad u_2^m = g \quad \text{on} \quad \Gamma_2, \quad u_2^m = \lambda_2^m \quad \text{on} \quad S_2, \\ \\ L_1^{(0)*} q_1^m = \chi_{12}(u_1^m - u_2^m) \quad \text{in} \quad \Omega_1, \quad (b_n^1)^+ q_1^m = 0 \quad \text{on} \quad \partial\Omega_1, \\ L_2^{(0)*} q_2^m = -\chi_{12}(u_1^m - u_2^m) \quad \text{in} \quad \Omega_2, \quad q_2^m = 0 \quad \text{on} \quad \partial\Omega_2, \\ \\ (b_n^1)^- \lambda_1^{m+1} = (b_n^1)^- \lambda_1^m - \gamma_m(\alpha (b_n^1)^- \lambda_1^m + (b_n^1)^- q_1^m) \quad \text{on} \quad S_1, \\ \lambda_2^{m+1} = \lambda_2^m - \gamma_m \left(\alpha \lambda_2^m - \nu \dfrac{\partial q_2^m}{\partial n} \right) \quad \text{on} \quad S_2, \\ v^{m+1} = v^m - \gamma_m(\alpha v^m + q_1^m) \text{ in } \Omega_{12} \end{array}\right. \tag{35}$$

The parameters $\{\gamma_m\}$ have to be chosen to get convergence of (35) (see [1, 14, 22]).

In the following we use the notion of "dense solvability" for problem (30),(31) (see [8]).

Definition 4.1 *We say that problem (30),(31) is densely solvable (or equivalently, that the property of dense solvability holds for (30),(31)) if for any* $\varepsilon_1 > 0$ *there exist* $\hat{\lambda}_1, \ \hat{\lambda}_2, \ \hat{v}$ *such that the problem*

$$\left\{\begin{array}{l} L_1 \hat{u}_1 = f + \omega \hat{v} \text{ in } \Omega_1, \\ \qquad (b_n^1)^- \hat{u}_1 = (b_n^1)^- g \text{ on } \Gamma_1, \quad (b_n^1)^- \hat{u}_1 = (b_n^1)^- \hat{\lambda}_1 \text{ on } S_1 \\ L_2 \hat{u}_2 = f \text{ in } \Omega_2, \quad \hat{u}_2 = g \text{ on } \Gamma_2, \quad \hat{u}_2 = \hat{\lambda}_2 \text{ on } S_2 \end{array}\right. \tag{36}$$

has solution $\hat{u}_1, \ \hat{u}_2$ *s.t.*

$$\|\hat{u}_1 - \hat{u}_2\|_{L_2(\Omega_{12})} \leq \varepsilon_1. \tag{37}$$

This is also referred to as a property of "approximate controllability" for problem (30),(31) (see [9], [23]).

As done in the previous section, we want to rewrite problem (30),(32) through linear operators. Therefore, we set $D(A) := \Lambda = L^2(S_1^-) \times H_{00}^{1/2}(S_2) \times L^2(\Omega_{12})$ and we define the linear operator

$$A : L^2(S_1^-) \times L^2(S_2) \times L^2(\Omega_{12}) \to L^2(\Omega_{12}), \quad A\tilde{\boldsymbol{\lambda}} := \chi_{12}(u_1^\lambda - u_2^\lambda), \tag{38}$$

where $\tilde{\boldsymbol{\lambda}} = (\tilde{\lambda}_1, \tilde{\lambda}_2, v)$ and where $u_1^\lambda, \ u_2^\lambda$ are the solutions of

$$\begin{array}{l} L_1 u_1^\lambda = \omega v \quad \text{in} \quad \Omega_1, \quad (b_n^1)^- u_1^\lambda = 0 \quad \text{on} \quad \Gamma_1, \quad (b_n^1)^- u_1^\lambda = (b_n^1)^- \tilde{\lambda}_1 \quad \text{on} \quad S_1, \\ L_2 u_2^\lambda = 0 \quad \text{in} \quad \Omega_2, \quad u_2^\lambda = 0 \quad \text{on} \quad \Gamma_2, \quad u_2^\lambda = \tilde{\lambda}_2 \quad \text{on} \quad S_2. \end{array} \tag{39}$$

The adjoint operator of A is

$$A^* : L^2(\Omega_{12}) \to L^2(S_1^-) \times L^2(S_2) \times L^2(\Omega_{12}), \quad A^* : w \mapsto \boldsymbol{\mu}, \tag{40}$$

such that:

$$\begin{aligned}
&L_1^{(0)*} q_1 = \chi_{12} w \ \text{ in } \ \Omega_1, \quad (b_n^1)^+ q_1 = 0 \ \text{ on } \ \partial\Omega_1, \\
&L_2^{(0)*} q_2 = -\chi_{12} w \ \text{ in } \ \Omega_2, \quad q_2 = 0 \ \text{ on } \ \partial\Omega_2, \\
&\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3), \quad \mu_1 = (b_n^1)^- q_{1|S_1}, \quad \mu_2 = -\nu \frac{\partial q_2}{\partial n}|_{S_2}, \quad \mu_3 = \omega v_{|\Omega_{12}}.
\end{aligned}$$

Given F as in (13), problem (30),(31) can be written again like (14), while problem (30),(32) can be written like (29).

Let us consider uniqueness of solution for problem (30),(31). Should two solutions $\{u_1^{(1)},\ u_2^{(1)},\ \lambda_1^{(1)}, \lambda_2^{(1)},\ v^{(1)}\}$ and $\{u_1^{(2)},\ u_2^{(2)},\ \lambda_1^{(2)},\ \lambda_2^{(2)},\ v^{(2)}\}$ exist, their difference $u_1 = u_1^{(1)} - u_1^{(2)}, \dots , v = v^{(1)} - v^{(2)}$ would satisfy the equations

$$\begin{cases}
L_1 u_1 = \omega v \ \text{ in } \ \Omega_1, \ (b_n^1)^- u_1 = 0 \ \text{ on } \ \Gamma_1, \ (b_n^1)^- u_1 = (b_n^1)^- \lambda_1 \ \text{ on } \ S_1, \\
L_2 u_2 = 0 \ \text{ in } \ \Omega_2, \quad u_2 = 0 \ \text{ on } \ \Gamma_2, \quad u_2 = \lambda_2 \ \text{ on } \ S_2, \\
u_1 = u_2, \ \text{ in } \ \Omega_{12}.
\end{cases} \tag{41}$$

From (41), for $u = u_1 = u_2$ in Ω_{12} we obtain the following boundary value problem

$$\begin{cases}
L_2 u = 0 \ \text{ in } \ \Omega_{12}, \\
L_1 u = b_n \dfrac{\partial u}{\partial n} + b_\tau \dfrac{\partial u}{\partial \tau} + \mu u = \omega v = 0 \ \text{ on } \ \partial\Omega_{12}.
\end{cases} \tag{42}$$

If the assumption (16) is fulfilled, then we have:

$$L_2 u = 0 \ \text{ in } \ \Omega_{12}, \quad u = \frac{\partial u}{\partial n} = 0 \ \text{ on } \ \Delta\Gamma^{(j)}, \ j = 1, \dots, p,$$

thus $u = 0$ in Ω_{12} from the uniqueness continuation theorem.
Hence, $v = 0, \ \lambda_k = 0, \ u_k = 0$ in $\Omega_k, \ k = 1, 2$, i.e. the uniqueness of solutions of (30),(31) takes place.
The same conclusion holds when assumption (17) or (18) hold instead of (16).
Moreover, the uniqueness of solution of (30),(31) implies the uniqueness of solution also for problem (34) (or equivalently (30),(32)) for $\alpha = 0$. The uniqueness results for (34) (or (30), (32)) when $\alpha > 0$ follow from well-posedeness of equation (29) [1].

Let us formulate the following proposition.

Proposition 4.1 *The following statements hold true:*

1. Problem (30),(31) is densely solvable.

2. For any $\alpha > 0$, problem (30),(32) has the unique solution $u_k = u_k(\alpha)$, $\lambda_k = \lambda_k(\alpha)$, $k = 1, 2$, $v = v(\alpha)$ and

$$\|\chi_{12}(u_1 - u_2)\|_{L_2(\Omega)} \to 0, \quad \alpha \to +0.$$

3. If problem (30),(31) has the unique solution $u_k^{(0)}, \lambda_k^{(0)}$, $k = 1, 2$, $v^{(0)}$, then

$$u_k(\alpha) \to u_k^{(0)}, \quad \lambda_k(\alpha) \to \lambda_k^{(0)}, \quad k = 1, 2, \quad v(\alpha) \to v^{(0)} \text{ as } \alpha \to +0,$$

where, for any $\alpha > 0$, $u_k(\alpha)$, $\lambda_k(\alpha)$, for $k = 1, 2$, $v(\alpha)$ is the unique solution of (30),(32).

4. If $\{u_k^m(\alpha)\}, \{\lambda_k^m(\alpha)\}$, $k = 1, 2$, $\{v^m(\alpha)\}$ are calculated by convergent iterative process (35) then for any $\varepsilon_2 > 0$ there are a small $\alpha > 0$ and sufficiently large $m = M \gg 1$ such that $\|\chi_{12}(u_1^M(\alpha) - u_2^M(\alpha))\|_{L_2(\Omega)} \le \varepsilon_2$, i.e. $u_k^M(\alpha), \lambda_k^M(\alpha)$, $k = 1, 2$, $v^M(\alpha)$ can be considered as an approximate solution of (30),(31).

5. If (30),(31) has the unique solution $u_k^{(0)}, \lambda_k^{(0)}$, $k = 1, 2$, $v^{(0)}$ then

$$u_k^m(\alpha) \to u_k^{(0)}, \quad \lambda_k^m(\alpha) \to \lambda_k^{(0)} (k = 1, 2), \quad v^m(\alpha) \to v^{(0)} \text{ as } \alpha \to 0 \text{ and } m \to \infty$$

and for sufficiently small $\alpha > 0$ and large $m = M \gg 1$ the functions $u_k^M(\alpha), \lambda_k^M(\alpha)$ (for $k = 1, 2$), $v^M(\alpha)$ can be chosen as approximations of $u_k^{(0)}, \lambda_k^{(0)}$ (for $k = 1, 2$), $v^{(0)}$.

Proof. *1.* Let us consider the homogeneous adjoint problem, corresponding to (30),(31): find q_1, q_2, w s.t.

$$\begin{cases} L_1^{(0)*} q_1 = \chi_{12} w \ \text{ in } \ \Omega_1, \quad (b_n^1)^+ q_1 = 0 \ \text{ on } \ \partial\Omega_1, \\ L_2^{(0)*} q_2 = -\chi_{12} w \ \text{ in } \ \Omega_2, \quad q_2 = 0 \ \text{ on } \ \partial\Omega_2, \\ (b_n^1)^- q_1 = 0 \ \text{ on } \ S_1, \quad -\nu \dfrac{\partial q_2}{\partial n} = 0 \ \text{ on } \ S_2, \quad \omega q_1 = 0 \text{ in } \Omega_1. \end{cases} \tag{43}$$

From the last relation we obtain: $q_1 = 0$ in Ω_{12}. Now, using the equations in Ω_1 and Ω_2 we conclude also that $w = 0$ in Ω_{12}, $q_k = 0$ in Ω_k, $k = 1, 2$. So, the adjoint problem (43) admits only the trivial solution, that is $ker(A^*) = \{0\}$. If we apply the theory of operator equations [8, 1], $ker(A^*) = \{0\}$ implies the dense solvability of (30),(31).

2. If $\alpha > 0$, existence and uniqueness of solution for problem (30),(32), (or equivalently (34)) is proved by invoking the results of [1]. As a matter of fact, if F is given as in (13), A as in (38) and A^* as in (40), problem (29) (or equivalently (30),(32)) has a unique solution $\boldsymbol{\lambda} = (\tilde{\lambda}_1, \tilde{\lambda}_2, v)$ for any $\alpha > 0$.
Besides, the solutions $u_k = u_k(\alpha)$, $\lambda_k = \lambda_k(\alpha)$, for $k = 1, 2$, $v = v(\alpha)$ of (30), (32), for sufficiently small $\alpha > 0$ can be chosen as "regularized approximations" of the solutions of (30),(31) such that (see [1]):

$$\begin{cases} L_1 u_1 = f + \omega v \ \text{ in } \ \Omega_1, \quad (b_n^1)^- u_1 = (b_n^1)^- g \ \text{ on } \ \Gamma_1, \quad (b_n^1)^- u_1 = (b_n^1)^- \lambda_1 \ \text{ on } \ S_1, \\ L_2 u_2 = f \ \text{ in } \ \Omega_2, \quad u_2 = g \ \text{ on } \ \Gamma_2, \quad u_2 = \lambda_2 \ \text{ on } \ S_2, \\ J_0(\lambda_1, \lambda_2) = \frac{1}{2}\|\chi_{12}(u_1(\lambda_1) - u_2(\lambda_2))\|^2_{L_2(\Omega)} \le \varepsilon_1 \end{cases} \tag{44}$$

and

$$\|\chi_{12}(u_1(\alpha) - u_2(\alpha))\|^2_{L_2(\Omega)} \to 0 \text{ as } \alpha \to +0. \tag{45}$$

3. If we apply the Tikhonov's regularization method to equation (29) (see [1, 20, 21]), we can prove the convergence

$$\boldsymbol{\lambda}(\alpha) \to \boldsymbol{\lambda}^{(0)} \qquad \text{as } \alpha \to +0.$$

Since the operator A is bounded and, by assumption, problem (30),(31) has a unique solution, then

$$u_k(\alpha) \to u_k^{(0)} \qquad \text{as } \alpha \to +0,\ k = 1, 2$$

as well.

4. This statement follows from both second assertion of the present Proposition and convergence of the iterative process (35) when the relaxation parameters γ_m are chosen appropriately (see [1, 14, 22]).

5. This statement is a consequence of the previous steps of the present proposition.
□

Remark 4.1 *Since $\|\chi_{12}(u_1 - u_2)\|^2_{L_2(\Omega)} \to 0$ as $\alpha \to +0$, the right hand side of the adjoint problem vanishes when $\alpha \to +0$, therefore we have the convergence result*

$$\|\mathbf{b} \cdot \nabla q_1\|_{L_2(\Omega_1)} + \|q_1\|_{L_2(\Omega_1)} + \|q_2\|_{L_2(\Omega_2)} \to 0,\ \|q_2\|_{H^1(\Omega_2)} \to 0, \quad \alpha \to +0$$

and

$$\|q_2\|_{H^2(\Omega_2)} \to 0,\ \alpha \to +0 \quad \textit{if } \Omega_2 \text{ is convex or } \partial\Omega_2 \text{ is smooth.}$$

Remark 4.2 *We draw attention to the following point: if $\|u_1 - u_2\|_{L_2(\Omega_{12})} \to 0,\ \alpha \to +0$ or $\|u_1^m - u_2^m\|_{L_2(\Omega_{12})} \to 0$ as "$\alpha \to +0,\ m \to \infty$" then we don't expect the convergence of both $v(\alpha)$ and $v^m(\alpha)$ to zero as $\alpha \to +0,\ m \to \infty$ in general case (because in this case it can be in contradiction with results from Propositions 3.1 - 3.2).*

5 Numerical results

We consider the heterogeneous problem (2) in the one-dimensional domain $\Omega = (0, 1)$, with $b_0 = 0$, $f = 1$, and homogeneous Dirichlet boundary conditions on $\partial\Omega$. The coefficients ν and b, as well as the subdomain partition will be specified below. For the 3-controls approach we chose $\omega \equiv \chi_{12}$ in Ω_{12}.
In order to discretize the differential problem, we consider a spectral element approach ([13]), where we denote by N the polynomial degree in each subdomain. After space discretization, both 2-controls approach (2),(4) and 3-controls approach (30),(32) are solved by Bi-CGStab iterations.

In Figure 4 we show the numerical solution obtained with both 2-controls (dashed line) and 3-controls (solid line), for $\nu = 1$, $b = 1$ at left and $\nu = 10^{-2}$,

$b = 1$ at right. For both cases the domain partition is $\Omega_1 = (0, 0.6)$ and $\Omega_2 = (0.3, 1)$. Note that, for the choice $b = 1$, we have $S_1^- = \emptyset$, thus no control function λ_1 is needed on S_1. The regularization parameter is $\alpha = 0$ in both cases, the same solution is obtained also for small $\alpha > 0$. We decompose both Ω_1 and Ω_2 in two spectral elements and the common element discretizes the overlap Ω_{12}. The polynomial degree used is $N = 16$ in each element of both Ω_1 and Ω_2 when $\nu = 1$, while it is $N = 16$ in each element of Ω_1 and $N = 24$ in $\Omega_2 \setminus \Omega_{12}$ when $\nu = 10^{-2}$. As we can see the solution obtained with 3-controls matches on the overlap Ω_{12} also with large viscosity $\nu = 1$. The dimension of system (29) is $n = 1$ for the 2 controls approach, while it is $n = N$ for the 3 controls approach.

In Figure 5 we show the numerical solution obtained with both 2 controls (dashed line) and 3 controls (solid line), for $\nu = 1$, $b = -1$ at left and $\nu = 10^{-2}$, $b = -1$ at right. The regularization parameter is $\alpha = 0$ in both cases, the same solution is obtained also for small $\alpha > 0$. The decomposition of Ω is the same used for the test case described in the previous figure. The polynomial degree used is $N = 16$ in each element of both Ω_1 and Ω_2 and for both $\nu = 1$ and $\nu = 10^{-2}$. In this case the dimension of system (29) is $n = 2$ for the 2 controls approach, while it is $n = N + 1$ for the 3 controls approach.

For simplicity of notation we set

$$\hat{J}_{2,\alpha} = \inf_{\lambda_1, \lambda_2} J_\alpha(\lambda_1, \lambda_1) \qquad \text{and} \qquad \hat{J}_{3,\alpha} = \inf_{\lambda_1, \lambda_2, v} J_\alpha(\lambda_1, \lambda_1, v). \tag{46}$$

From Figure 6 we note that both $\hat{J}_{2,\alpha}$ and $\hat{J}_{3,\alpha}$ vanish as the viscosity tends to zero. But, as shown in Figure 7 for fixed viscosity and for increasing polynomial degree N, the values $\hat{J}_{2,\alpha}$ is positive and bounded from below, while the value $\hat{J}_{3,\alpha}$ tends to zero.

Finally, in Figure 8 both $\hat{J}_{2,\alpha}$ and $\hat{J}_{3,\alpha}$ are plotted versus the size of the overlap.

We consider now some two-dimensional cases.

Test case #1. We consider the following data:

$$\Omega = (-1, 1)^2, \ \Omega_1 = (-1, .8) \times (-1, 1), \ \Omega_2 = (.7, 1) \times (-1, 1), \tag{47}$$

$$\vec{b} = [y, 0]^t, \ b_0 = 1, \ f \equiv 1. \tag{48}$$

We impose homogeneous Dirichlet conditions on the right vertical side of Ω, $g \equiv 1$ on $\{-1\} \times (0, 1]$ and null normal derivative on the horizontal sides of Ω and on $\{-1\} \times [-1, 0]$. In this case we have $S_1^- = \{.8\} \times [-1, 0)$. Along the y coordinate the mesh is uniform, while along the x coordinate the mesh is finer near the boundary layer, in particular we have used polynomial degree $N = 5$ along the $y-$ direction and $N = 12$ along the $x-$ direction. Ω_1 is decomposed in 2×3 spectral elements, while Ω_2 is decomposed in 4×3 spectral elements. The numerical solution obtained with viscosity $\nu = 10^{-2}$ and $\nu = 10^{-3}$ for the

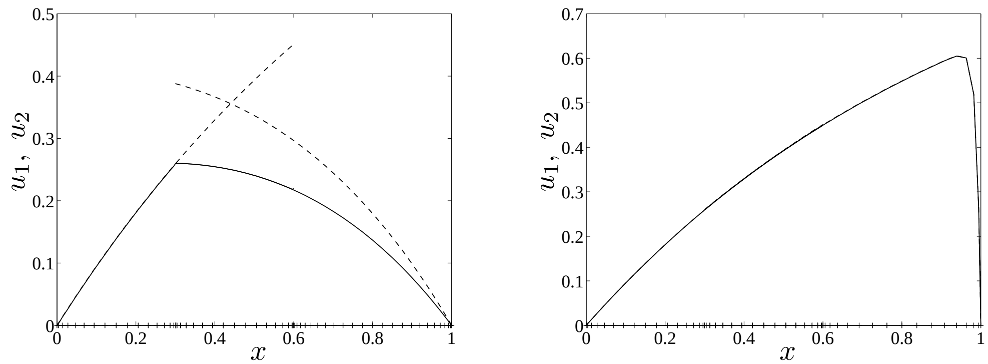

Figure 4: Numerical solutions obtained with 2 controls (dashed line) and 3 controls (solid line) for $\nu = 1$, $b = 1$ at left and for $\nu = 10^{-2}$, $b = 1$ at right. $\Omega_1 = (0, 0.6)$, $\Omega_2 = (0.3, 1)$.

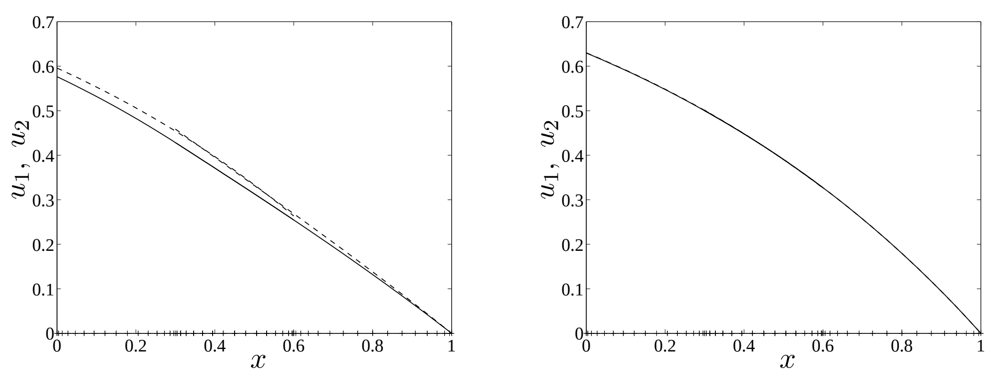

Figure 5: Numerical solutions obtained with 2 controls (dashed line) and 3 controls (solid line) for $\nu = 1$, $b = -1$ at left and for $\nu = 10^{-2}$, $b = -1$ at right. $\Omega_1 = (0, 0.6)$, $\Omega_2 = (0.3, 1)$.

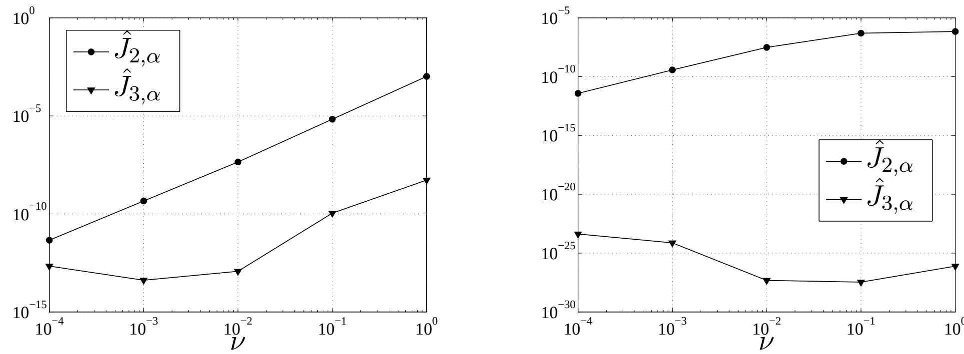

Figure 6: The minimum values $\hat{J}_{2,\alpha}$ and $\hat{J}_{3,\alpha}$ versus the viscosity ν, obtained with $b = 1$ at left and $b = -1$ at right. For both cases, $\alpha = 0$ and $\Omega_1 = (0, 0.6)$, $\Omega_2 = (0.3, 1)$. The polynomial degree N is chosen inside the spectral elements in order to guarantee absence of oscillations. Ω is decomposed in 3 spectral elements as for the test cases presented in the previous pictures.

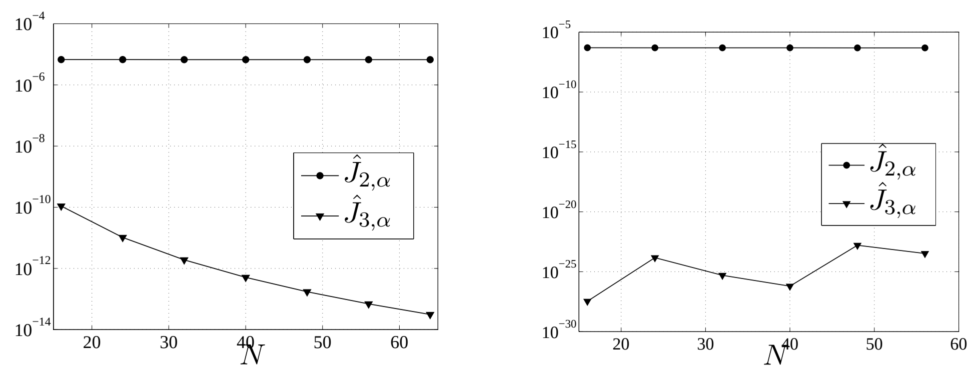

Figure 7: The minimum values $\hat{J}_{2,\alpha}$ and $\hat{J}_{3,\alpha}$ versus the polynomial degree N, obtained with $b = 1$ at left and $b = -1$ at right. For both cases, $\nu = 0.1$, $\alpha = 0$ and $\Omega_1 = (0, 0.6)$, $\Omega_2 = (0.3, 1)$. Ω is decomposed in 3 spectral elements as for the test cases presented in the previous pictures.

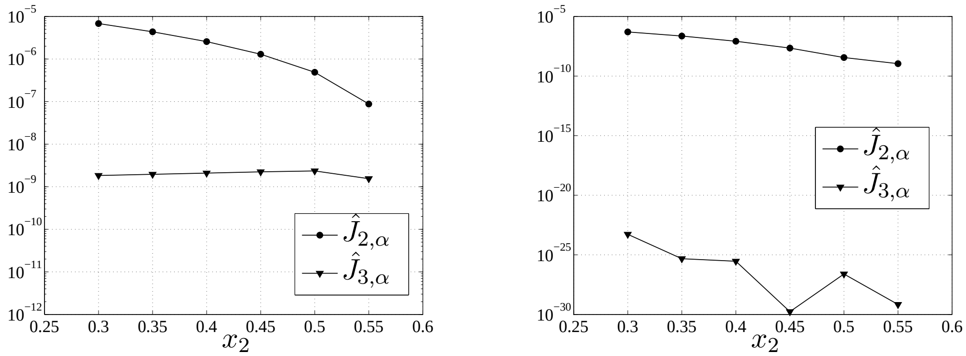

Figure 8: The minimum values $\hat{J}_{2,\alpha}$ and $\hat{J}_{3,\alpha}$ versus the size of the overlap, obtained with $\nu = 0.1$, $b = 1$ at left and $\nu = 0.01$, $b = -1$ at right. For both cases $\alpha = 0$, we have decomposed Ω in 20 equal spectral elements of equal size $H = 0.5$ and $N = 4$ in each element, $\Omega_1 = (0, 0.6)$ and $\Omega_2 = (x_2, 1)$. The size of the overlap is $meas(\Omega_{12}) = 0.6 - x_2$.

2-controls approach is shown in Figure 9, while in Figure 10 we report the solution of the 3-controls approach.
The numerical solution of the minimum problem is computed by BiCG-Stab iterations. The stopping criterion is that the norm of the normalized residual be below a tolerance $\varepsilon = 10^{-6}$. In Table 1 we show the number of iterations needed to solve the minimum problem for both 2-controls and 3-controls approaches, as well as the minimum values $\hat{J}_{2,\alpha}(\lambda_1, \lambda_2)$ and $\hat{J}_{3,\alpha}(\lambda_1, \lambda_2, v)$, defined in (46), versus the viscosity ν. We may observe that, for the 2-controls approach, the number of Bi-CGStab iterations is independent of the viscosity, while $\hat{J}_{2,\alpha}(\lambda_1, \lambda_2)$ decreases for vanishing viscosities. On the other hand, the number of BiCG-Stab iterations required by the 3-controls approach decreases when the viscosity vanishes and it is considerably larger than for the 2-controls approach. Note that the dimension n of system (29) for the 3-controls approach is larger than for the 2-controls approach. As a matter of fact, the third control v is defined on the whole overlapping region Ω_{12}. For the discretization used in this test case, we have $n = 24$ for the 2-controls approach and $n = 193$ for the 3-controls approach. The minimum values $\hat{J}_{3,\alpha}(\lambda_1, \lambda_2, v)$ do not depend on the viscosity. We remark that when we impose a more restrictive tolerance for the stopping criterion of BiCG-Stab iterations (say $\varepsilon = 10^{-12}$), the minimum values $\hat{J}_{3,\alpha}(\lambda_1, \lambda_2, v)$ are about 10^{-24} for any value of ν, confirming the second assertion of Proposition 4.1.
In the case of the 3-controls approach, if we replace the BiCG-Stab stopping criterion on the residual with $J_\alpha(\lambda_1, \lambda_2, v) \leq \hat{J}_{2,\alpha}(\lambda_1, \lambda_2)$, where $\hat{J}_{2,\alpha}(\lambda_1, \lambda_2)$ is the minimum obtained for the same value of the viscosity by the 2-controls approach, then the numerical solution obtained is very poorly resolved and presents

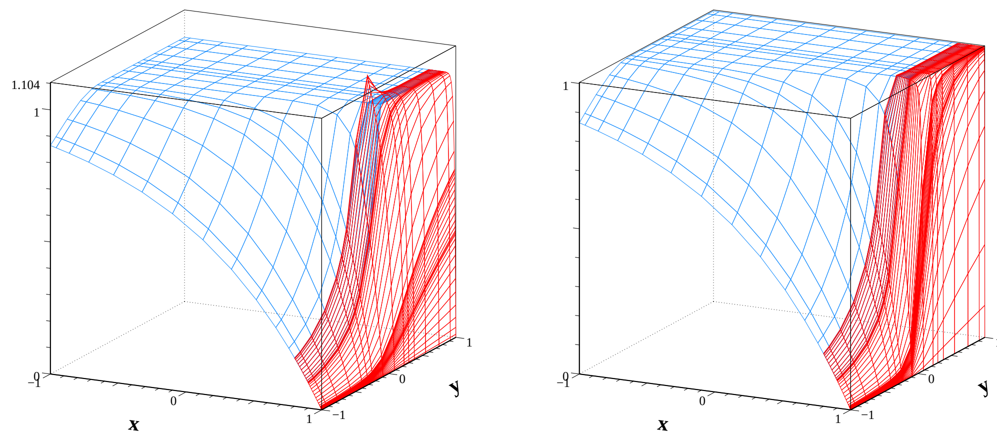

Figure 9: Test case #1. Numerical solution for $\nu = 10^{-2}$ (left) and $\nu = 10^{-3}$ (right) obtained through the 2-controls approach with $\alpha = 0$. The overlap is $\Omega_{12} = (.7, .8) \times (-1, 1)$. The minimum values of the cost functional J are $\hat{J}_{2,\alpha} = 5.67 \cdot 10^{-5}$ for $\nu = 10^{-2}$ and $\hat{J}_{2,\alpha} = 4.92 \cdot 10^{-7}$ for $\nu = 10^{-3}$.

	2-controls		3-controls	
ν	#it	$\hat{J}_{2,\alpha}$	#it	$\hat{J}_{3,\alpha}$
0.1	18	$8.71 \cdot 10^{-4}$	319	$2.83 \cdot 10^{-11}$
0.01	15	$5.85 \cdot 10^{-5}$	276	$1.97 \cdot 10^{-11}$
0.001	18	$4.92 \cdot 10^{-7}$	220	$5.81 \cdot 10^{-11}$
0.0001	18	$9.79 \cdot 10^{-9}$	190	$2.45 \cdot 10^{-11}$

Table 1: Number of BiCG-Stab iterations needed to satisfy the stopping criterion on the residual with tolerance $\varepsilon = 10^{-6}$ and the obtained minimum values for the cost functionals $J_2(\lambda_1, \lambda_2)$ and $J_3(\lambda_1, \lambda_2, v)$.

oscillations inside the domain Ω_1.

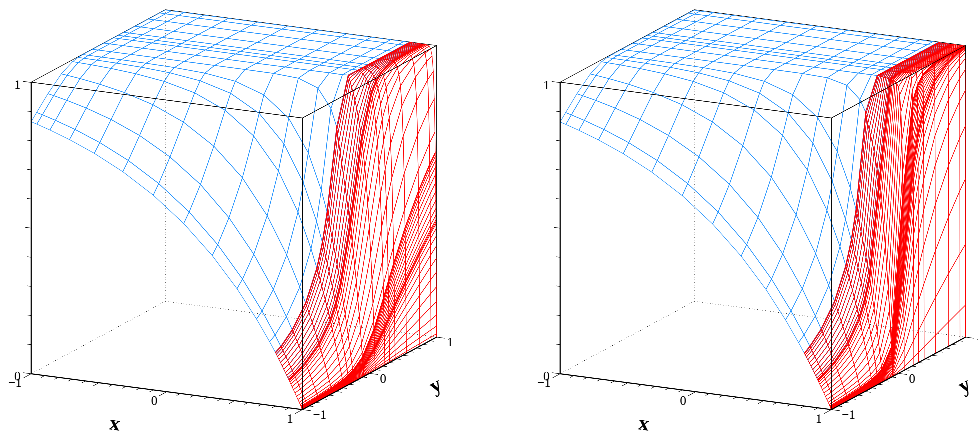

Figure 10: Test case #1. Numerical solution for $\nu = 10^{-2}$ (left) and $\nu = 10^{-3}$ (right) obtained through the 3-controls approach with $\alpha = 0$. The overlap is $\Omega_{12} = (.7, .8) \times (-1, 1)$. The minimum values of the cost functional J are $\hat{J}_{3,\alpha} = 1.97 \cdot 10^{-11}$ for $\nu = 10^{-2}$ and $\hat{J}_{3,\alpha} = 5.81 \cdot 10^{-11}$ for $\nu = 10^{-3}$.

Test case #2. We consider now a rectangular domain Ω with two circular holes, as described in Figure 11, and the following data

$$\vec{b} = [1, 0]^t, \; b_0 = 10^{-1}, \; f \equiv 0. \tag{49}$$

The solution is given $u = 1$ on the left vertical side of Ω, that is the inflow external boundary for Ω_1, while it is $u = 0$ on the boundaries of the holes.

The space discretization is performed with conformal quadrilateral spectral elements with polynomial degree $N = 8$ in each direction. The numerical solution obtained through the 3-controls approach, for $\nu = 0.05$, is shown in Figure 12.

6 Domain decomposition algorithms with "mixed - type" controls

We investigate domain decomposition algorithms based on optimal control approaches with different types of controls on S_1^-, S_2^- and in Ω_{12}. We consider the following optimal control problem: find $u_1, u_2, \lambda_1, \lambda_2, v$ s.t.

$$\begin{aligned}
&L_1 u_1 = f + \omega v \quad \text{in} \quad \Omega_1, \\
&\qquad (b_n^1)^- u_1 = (b_n^1)^- g \quad \text{on} \quad \Gamma_1, \quad (b_n^1)^- u_1 = (b_n^1)^- \lambda_1 \quad \text{on} \quad S_1, \\
&L_2 u_2 = f \quad \text{in} \quad \Omega_2, \\
&\qquad u_2 = g \quad \text{on} \quad \Gamma_2, \quad \left(\nu \frac{\partial u_2}{\partial n} + (b_n^2)^- u_2 \right) = (b_n^2)^- \lambda_2 \quad \text{on} \quad S_2,
\end{aligned} \tag{50}$$

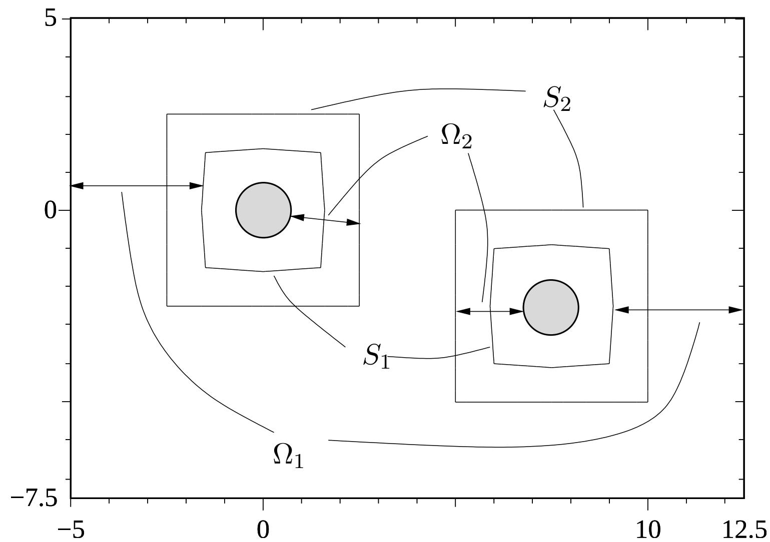

Figure 11: The computational domain for test case #2.

$$\inf_{\lambda_1,\lambda_2,v} J_\alpha(u_1,u_2,\lambda_1,\lambda_2,v), \tag{51}$$

where ω and J_α have been defined in the previous section, with the exception that now the term $\int_{S_2} \lambda_2^2 d\Gamma$ is replaced by $\int_{S_2} (b_n^2)^- \lambda_2^2 d\Gamma$.

The variational equations corresponding to (51) are

$$\begin{cases} L_1^{(0)*} q_1 = \chi_{12}(u_1 - u_2) \ \text{ in } \ \Omega_1, \quad (b_n^1)^+ q_1 = 0 \ \text{ on } \ \partial\Omega_1, \\ L_2^{(0)*} q_2 = -\chi_{12}(u_1 - u_2) \ \text{ in } \ \Omega_2, \\ \qquad q_2 = 0 \ \text{ on } \ \Gamma_2, \quad \left(\nu \dfrac{\partial q_2}{\partial n} + (b_n^2)^+ q_2\right) = 0 \ \text{ on } \ S_2, \\ \alpha(b_n^1)^- \lambda_1 + (b_n^1)^- q_1 = 0 \ \text{ on } \ S_1, \quad \alpha(b_n^2)^- \lambda_2 + (b_n^2)^- q_2 = 0 \ \text{ on } \ S_2, \\ \alpha\omega v + \omega q_1 = 0 \ \text{ on } \ \Omega_1. \end{cases} \tag{52}$$

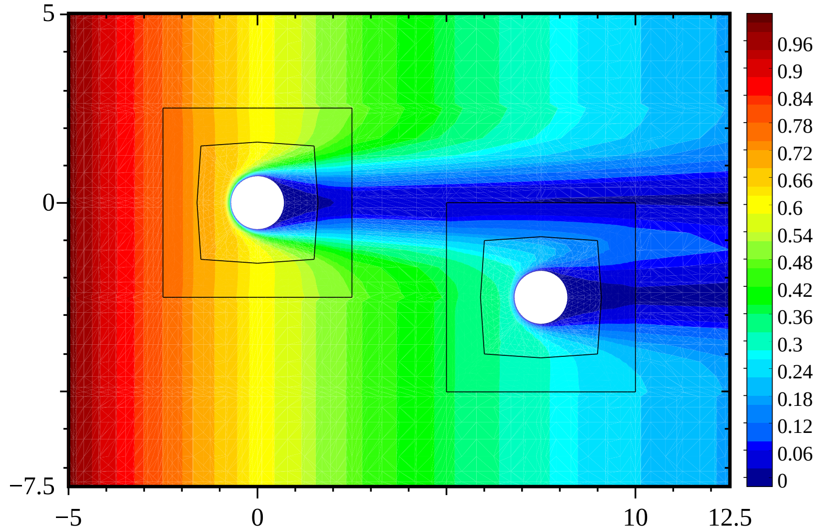

Figure 12: Test case #2. The 3-controls solution when $\nu = 0.05$. No regularization is done on the problem, i.e. $\alpha = 0$. The interfaces S_1 and S_2 are drawn over the solution.

The iterative process that we propose to solve (52), is: for any given $\lambda_1^0,\ \lambda_2^0$,

$$\left\{\begin{array}{l} L_1 u_1^m = f + \omega v^m \;\text{ in }\; \Omega_1, \\ \qquad (b_n^1)^- u_1^m = (b_n^1)^- g \;\text{ on }\; \Gamma_1, \quad (b_n^1)^- u_1^m = (b_n^1)^- \lambda_1^m \;\text{ on }\; S_1, \\ L_2 u_2^m = f \;\text{ in }\; \Omega_2, \\ \qquad u_2^m = g, \quad \left(\nu \dfrac{\partial u_2^m}{\partial n} + (b_n^2)^- u_2^m\right) = (b_n^2)^- \lambda_2^m \;\text{ on }\; S_2, \\ L_1^{(0)*} q_1^m = \chi_{12}(u_1^m - u_2^m) \;\text{ in }\; \Omega_1, \quad (b_n^1)^+ q_1^m = 0 \;\text{ on }\; \partial\Omega_1, \\ L_2^{(0)*} q_2^m = -\chi_{12}(u_1^m - u_2^m) \;\text{ in }\; \Omega_2, \\ \qquad q_2^m = 0 \;\text{ on }\; \Gamma_2, \quad \left(\nu \dfrac{\partial q_2^m}{\partial n} + (b_n^2)^+ q_2^m\right) = 0 \;\text{ on }\; S_2, \\ (b_n^1)^- \lambda_1^{m+1} = (b_n^1)^- \lambda_1^m - \gamma_m(\alpha (b_n^1)^- \lambda_1^m + (b_n^1)^- q_1^m) \;\text{ on }\; S_1, \\ (b_n^2)^- \lambda_2^{m+1} = (b_n^2)^- \lambda_2^m - \gamma_m(\alpha (b_n^2)^- q_2^m + (b_n^2)^- q_2^m) \;\text{ on }\; S_2, \\ \omega v^{m+1} = \omega v^m - \gamma_m(\alpha \omega v^m + \omega q_1^m) \;\text{ in }\; \Omega_1, \; m = 0, 1, \ldots. \end{array}\right. \tag{53}$$

Proposition 6.1 *The assertions of Proposition 4.1 hold true for both problem (50),(31) and (50),(51) (instead of (30),(31) and (30),(32), respectively) and for the process (53) (instead of (35)).*

Proof. We have to prove that problem (50),(31) is densely solvable. Let us consider the adjoint problem, find $q_1,\ q_2,\ w$ such that

$$\left\{\begin{array}{l} L_1^{(0)*}q_1 = \chi_{12}w \ \text{ in } \ \Omega_1, \quad (b_n^1)^+q_1 = 0 \ \text{ on } \ \partial\Omega_1, \\ L_2^{(0)*}q_2 = -\chi_{12}w \ \text{ in } \ \Omega_2, \\ \qquad q_2 = 0 \ \text{ on } \ \Gamma_2, \quad \left(\nu \dfrac{\partial q_2}{\partial n} + (b_n^2)^+q_2\right) = 0 \ \text{ on } \ S_2, \\ (b_n^1)^-q_1 = 0 \ \text{ on } \ S_1, \quad (b_n^2)^-q_2 = 0 \ \text{ on } \ S_2, \quad \omega q_1 = 0 \ \text{ in } \ \Omega_1. \end{array}\right. \tag{54}$$

The latter relation implies that $q_1 = 0$ in Ω_1. Therefore, $w = 0$ in Ω_{12} and the function q_2 satisfies the following equations:

$$L_2^{(0)*}q_2 = 0 \ \text{ in } \ \Omega_2, \quad q_2 = 0 \ \text{ on } \ \Gamma_2, \quad \left(\nu \frac{\partial q_2}{\partial n} + (b_n^2)^+q_2\right) = 0 \ \text{ on } \ S_2,$$
$$(b_n^2)^-q_2 = 0 \ \text{ on } \ S_2.$$

Hence, if $q_2 = 0$ in Ω_2, problem (54) has the trivial solution, and we conclude that the boundary value problem (50),(31) is densely solvable.

The other steps of the proof can be carried out following the proof of Proposition 4.1. □

7 Domain decomposition algorithm for the second order elliptic equations

We revisit the control approach developed in the previous sections for heterogeneous domain decomposition methods in the case of a "standard" (homogeneous) domain decomposition method for elliptic equations.

Let us consider in $\Omega \subset \mathbb{R}^2$ the Dirichlet problem for the second order equation given by

$$Lu := -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + \nabla\cdot(\mathbf{b}u) + b_0u = f \ \text{ in } \Omega, \qquad u = 0 \ \text{ on } \partial\Omega, \tag{55}$$

where a_{ij} are bounded smooth functions such that

$$\sum_{i,j=1}^{2} a_{ij}\xi_i\xi_j \geq C_0|\xi|^2 \ \forall\xi \in \mathbb{R}^2, \quad x \equiv (x_1, x_2) \in \overline{\Omega}$$

with $C_0 = \text{const} > 0$ and conditions on $\mathbf{b}, b_0$ as before.

We introduce the decomposition of Ω onto two overlapping subsets as in Figs. 1 - 3. To simplify our considerations we assume each subdomain $\Omega_{12}^{(j)}$ to be convex or $\partial\Omega_{12}^{(j)}$ is smooth, for $j = 1, ..., p$. Let us consider the "exact controllability problem": find $u_1,\ u_2,\ \lambda_1,\ \lambda_2,\ v$ such that

$$\left\{\begin{array}{ll} Lu_1 = f + \chi_{12} v \ \text{ in } \Omega_1, & u_1 = 0 \ \text{ on } \Gamma_1, \\ & \left(\dfrac{\partial u_1}{\partial n_L} + (b_n^1)^- u_1\right) = (b_n^1)^- \lambda_1 \ \text{ on } S_1, \\ Lu_2 = f \ \text{ in } \Omega_2, & u_2 = 0 \ \text{ on } \Gamma_2, \\ & \left(\dfrac{\partial u_2}{\partial n_L} + (b_n^2)^- u_2\right) = (b_n^2)^- \lambda_2 \ \text{ on } S_2, \\ u_1 = u_2 \ \text{ in } \Omega_{12}, & \end{array}\right. \tag{56}$$

where $\partial u/\partial n_L = \sum\limits_{i,j=1}^{2} a_{ij} n_i \partial u/\partial x_j$, $\mathbf{n} = (n_1, n_2)$ is the outward unit normal vector on the boundary. The weak statement of (56) reads: find $u_k \in H^1_{\Gamma_k}(\Omega_k)$, $\lambda_k \in L_2(S_k^-)$, $v \in L_2(\Omega_{12})$ s.t.

$$\left\{\begin{array}{ll} a_1(u_1, \hat{u}_1) = (f, \hat{u}_1)_{L_2(\Omega_1)} + \displaystyle\int_{S_1} (b_n^1)^- \lambda_1 \hat{u}_1 d\Gamma + (\chi_{12} v, \hat{u}_1)_{L_2(\Omega_1)} & \forall \hat{u}_1 \in H^1_{\Gamma_1}(\Omega_1), \\ a_2(u_2, \hat{u}_2) = (f, \hat{u}_2)_{L_2(\Omega_2)} + \displaystyle\int_{S_2} (b_n^2)^- \lambda_2 \hat{u}_2 d\Gamma & \forall \hat{u}_2 \in H^1_{\Gamma_2}(\Omega_2), \\ J_0(u_1, u_2) := \dfrac{1}{2} \|\chi_{12}(u_1 - u_2)\|^2_{L_2(\Omega)} = 0 & \end{array}\right. \tag{57}$$

where

$$H^1_{\Gamma_k}(\Omega_k) = \{u \in H^1(\Omega_k), \quad u = 0 \ \text{ on } \Gamma_k\},$$

$$\begin{aligned} a_k(u_k, \hat{u}_k) = & \int_{\Omega_k} \left(\sum_{i,j=1}^{2} a_{ij} \frac{\partial u_k}{\partial \xi_j} \frac{\partial \hat{u}_k}{\partial \xi_i} - u_k \mathbf{b} \cdot \nabla \hat{u}_k + b_0 u_k \hat{u}_k \right) d\Omega \\ & + \int_{S_k} (b_n^k)^+ u_k \hat{u}_k d\Gamma, \ k = 1, 2. \end{aligned} \tag{58}$$

The optimal control problem reads as follows: find $u_k = u_k(\alpha) \in H^1_{\Gamma_k}(\Omega_k)$, $\lambda_k = \lambda_k(\alpha) \in L_2(S_k^-)$, $k = 1, 2$ and $v = v(\alpha) \in L_2(\Omega_{12})$ s.t.

$$\left\{\begin{array}{l} a_1(u_1, \hat{u}_1) = (f, \hat{u}_1)_{L_2(\Omega_1)} + \displaystyle\int_{S_1} (b_n^1)^- \lambda_1 \hat{u}_1 d\Gamma + (\chi_{12} v, \hat{u}_1)_{L_2(\Omega_1)} \\ \hfill \forall \hat{u}_1 \in H^1_{\Gamma_1}(\Omega_1), \\ a_2(u_2, \hat{u}_2) = (f, \hat{u}_2)_{L_2(\Omega_2)} + \displaystyle\int_{S_2} (b_n^2)^- \lambda_2 \hat{u}_2 d\Gamma \quad \forall \hat{u}_2 \in H^1_{\Gamma_2}(\Omega_2), \\ \inf\limits_{\lambda_1, \lambda_2, v} J_\alpha(\lambda_1, \lambda_2, v), \end{array}\right. \tag{59}$$

where

$$J_\alpha(\lambda_1,\lambda_2,v) = \frac{1}{2}\alpha\int\limits_{S_1}(b_n^1)^-\lambda_1^2 d\Gamma + \frac{1}{2}\alpha\int\limits_{S_2}(b_n^2)^-\lambda_2^2 d\Gamma + \frac{1}{2}\alpha\int\limits_{\Omega_{12}}\chi_{12}v^2 d\Omega + \frac{1}{2}\|\chi_{12}(u_1-u_2)\|^2_{L_2(\Omega)}.$$

The variational equations corresponding to (59) are:

$$\begin{cases} a_1(u_1,\hat{u}_1) = (f,\hat{u}_1)_{L_2(\Omega_1)} + \int\limits_{S_1}(b_n^1)^-\lambda_1\hat{u}_1 d\Gamma + (\chi_{12}v,\hat{u}_1)_{L_2(\Omega_1)} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall\hat{u}_1 \in H^1_{\Gamma_1}(\Omega_1), \\ a_2(u_1,\hat{u}_2) = (f,\hat{u}_2)_{L_2(\Omega_2)} + \int\limits_{S_2}(b_n^2)^-\lambda_2\hat{u}_2 d\Gamma \quad \forall\hat{u}_2 \in H^1_{\Gamma_2}(\Omega_2), \\ a_1(\hat{q}_1,q_1) = (\chi_{12}(u_1-u_2),\hat{q}_1)_{L_2(\Omega_1)} \qquad \forall\hat{q}_1 \in H^1_{\Gamma_1}(\Omega_1), \\ a_2(\hat{q}_2,q_2) = -(\chi_{12}(u_1-u_2),\hat{q}_2)_{L_2(\Omega_1)} \qquad \forall\hat{q}_2 \in H^1_{\Gamma_2}(\Omega_2), \\ \alpha(b_n^1)^-\lambda_1 + (b_n^1)^-q_1 = 0 \quad \text{a. e. on } S_1, \\ \alpha(b_n^2)^-\lambda_2 + (b_n^2)^-q_2 = 0 \quad \text{a. e. on } S_2, \\ \alpha\chi_{12}v + \chi_{12}q_1 = 0 \quad \text{in } \Omega_1. \end{cases} \tag{60}$$

The following iterative process is similar to those proposed in the previous sections: if $\lambda_1^0,\ \lambda_2^0,\ v^0$ are given, then the functions $\lambda_1^{m+1}, \lambda_2^{m+1}, v^{m+1}$ are obtained from the solution of:

$$\begin{cases} a_1(u_1^m,\hat{u}_1) = (f,\hat{u}_1)_{L_2(\Omega_1)} + \int\limits_{S_1}(b_n^1)^-\lambda_1^m\hat{u}_1 d\Gamma + (\chi_{12}v^m,\hat{u}_1)_{L_2(\Omega_1)} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall\hat{u}_1 \in H^1_{\Gamma_1}(\Omega_1), \\ a_2(u_2^m,\hat{u}_2) = (f,\hat{u}_2)_{L_2(\Omega_2)} + \int\limits_{S_2}(b_n^2)^-\lambda_2^m\hat{u}_2 d\Gamma \quad \forall\hat{u}_2 \in H^1_{\Gamma_2}(\Omega_2), \\ a_1(\hat{q}_1,q_1^m) = (\chi_{12}(u_1^m-u_2^m),\hat{q}_1)_{L_2(\Omega_1)} \qquad \forall\hat{q}_1 \in H^1_{\Gamma_1}(\Omega_1), \\ a_2(\hat{q}_2,q_2^m) = -(\chi_{12}(u_1^m-u_2^m),\hat{q}_2)_{L_2(\Omega_2)} \qquad \forall\hat{q}_2 \in H^1_{\Gamma_2}(\Omega_2), \\ (b_n^1)^-\lambda_1^{m+1} = (b_n^1)^-\lambda_1^m - \gamma_m(\alpha(b_n^1)^-\lambda_1^m + (b_n^1)^-q_1^m) \quad \text{on } S_1, \\ (b_n^2)^-\lambda_2^{m+1} = (b_n^2)^-\lambda_2^m - \gamma_m(\alpha(b_n^2)^-\lambda_2^m + (b_n^2)^-q_2^m) \quad \text{on } S_2, \\ \chi_{12}v^{m+1} = \chi_{12}v^m - \gamma_m(\alpha\chi_{12}v^m + \chi_{12}q_1^m) \quad \text{on } \Omega_1, \end{cases} \tag{61}$$

for $m = 0,\ 1,\dots$ and for suitable parameters $\{\gamma_m\}$.

Let us analyze the convergence of solutions given by (61) to the solution of (55) as $\alpha \to +0$ and $m \to \infty$. The following statement holds true.

Proposition 7.1 *Problem (56) (or equivalently (57)) as a unique solution and it is densely solvable. Moreover, if* $u^{(0)}$ *is the solution of (55), and* $u_k^{(0)} \equiv u^{(0)}$ *in* Ω_k, $u_k^m(\alpha)$, $\lambda_k^m(\alpha)$, *(for* $k = 1, 2$*)* $v^m(\alpha)$ *is the solution obtained by the iterative algorithm (61), then*

$$\sum_{k=1}^{2} \|u_k^{(0)} - u_k^m(\alpha)\|_{H^1(\Omega_k)} \to 0 \quad \text{as} \quad \alpha \to +0, \quad m \to \infty. \tag{62}$$

Proof. Under the assumptions imposed on $\mathbf{b}$, b_0, f, Ω in Section 2, problem (55) has a unique solution $u \in H_0^1(\Omega)$. Hence the same function u is also solution of problem (56) (or equivalently (57)) when $v \equiv 0$. Assume that (56) has another solution $\tilde{u}$. Then the difference $\hat{u} = u - \tilde{u}$ satisfies the following equations

$$\begin{cases} L\hat{u}_1 = \chi_{12} v \text{ in } \Omega_1, \quad \hat{u}_1 = 0 \text{ on } \Gamma_1, \quad \left(\dfrac{\partial \hat{u}_1}{\partial n_L} + (b_n^1)^- \hat{u}_1\right) = (b_n^1)^- \lambda_1 \text{ on } S_1, \\ L\hat{u}_2 = 0 \text{ in } \Omega_2, \quad \hat{u}_2 = 0 \text{ on } \Gamma_2, \quad \left(\dfrac{\partial \hat{u}_2}{\partial n_L} + (b_n^2)^- \hat{u}_2\right) = (b_n^2)^- \lambda_2 \text{ on } S_2, \\ \hat{u}_1 = \hat{u}_2 \text{ in } \Omega_{12}. \end{cases}$$

Since the differential operators in the first and second equations coincide, then using the latter equality we conclude: $v = 0$ in Ω_{12}. (This conclusion can be obtained also from the weak statement (57) of problem (56)). Now, let us consider the following integral

$$I(u, \hat{u}) \equiv \int_\Omega \left(\sum_{i,j=1}^{2} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial \hat{u}}{\partial x_j} - u\mathbf{b} \cdot \nabla \hat{u} + b_0 u \hat{u} \right) d\Omega \qquad \forall \hat{u} \in H_0^1(\Omega),$$

where $u_k = u_{|\Omega_k}$ for $k = 1, 2$ and $u_1 = u_2 = u$ in Ω_{12}. Integrating by parts we have:

$$I(u, \hat{u}) = (Lu_1, \hat{u})_{L_2(\Omega_1 \setminus \Omega_{12})} + (Lu_2, \hat{u})_{L_2(\Omega_2 \setminus \Omega_{12})} + (Lu, \hat{u})_{L_2(\Omega_{12})} +$$

$$+ \int_{S_1} \left(-\frac{\partial u_2}{\partial n_L} + \frac{\partial u}{\partial n_L} \right) \hat{u} d\Gamma + \int_{S_2} \left(-\frac{\partial u_1}{\partial n_L} + \frac{\partial u}{\partial n_L} \right) \hat{u} d\Gamma,$$

where $Lu_k = 0$ in $\Omega_k \setminus \Omega_{12}, k = 1, 2$ and $Lu = 0$ in Ω_{12}. Since

$$u_1 = u_2 = u \text{ in } \Omega_{12}, \qquad \frac{\partial u}{\partial n_L} = \frac{\partial u_1}{\partial n_L} \text{ on } S_2, \qquad \frac{\partial u}{\partial n_L} = \frac{\partial u_2}{\partial n_L} \text{ on } S_1,$$

then $I(u, \hat{u}) = 0$. If we set $\hat{u} = u$, then

$$I(u, u) = \int_\Omega \left(\sum_{i,j=1}^{2} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + \left(b_0 + \frac{1}{2} \text{div} \mathbf{b} \right) u^2 \right) d\Omega = 0$$

and $u = 0$ in Ω, $u_k = 0$ in Ω_k, $k = 1, 2$, $\lambda_1 = 0$ on S_1^-, $\lambda_2 = 0$ on S_2^- i.e. the solution of (56) (or equivalently (57)) is unique.

Let us consider now the weak statement of the adjoint problem with homogeneous boundary conditions: find q_1, q_2, w s.t.

$$\begin{cases} a_1(\hat{q}_1, q_1) = (\chi_{12} w, \hat{q}_1)_{L_2(\Omega_1)} & \forall \hat{q}_1 \in H_{\Gamma_1}^1(\Omega_1), \\ a_2(\hat{q}_2, q_2) = -(\chi_{12} w, \hat{q}_2)_{L_2(\Omega_2)} & \forall \hat{q}_2 \in H_{\Gamma_2}^1(\Omega_2), \\ (b_n^1)^- q_1 = 0 \text{ a. e. on } S_1, \quad (b_n^2)^- q_2 = 0 \text{ a. e. on } S_2, \quad \chi_{12} q_1 = 0 \text{ in } \Omega_1. \end{cases} \tag{63}$$

Since $\chi_{12}q_1 = 0$ in Ω_1 then $q_1 = 0$ in Ω_{12} and $w = 0$. If we set $\hat{q}_2 = q_2$ then:

$$a_2(q_2, q_2) = \int\limits_{\Omega_2} \left(\sum_{i,j=1}^{2} a_{ij} \frac{\partial q_2}{\partial x_i} \frac{\partial q_2}{\partial x_j} + \left(b_0 + \frac{1}{2}\mathrm{div}\mathbf{b} \right) q_2^2 \right) d\Omega + \frac{1}{2} \int\limits_{S_2} (b_n^2)^+ q_2^2 d\Gamma = 0$$

and $q_2 = 0$ in Ω_2, $q_1 = 0$ in Ω_1, $w = 0$, i.e. problem (61) has the trivial solution. This means that problem (56) (or equivalently (57)) is densely solvable and the following relation holds true (see [1]):

$$\sum_{k=1}^{2} \left(\|u_k^{(0)} - u_k(\alpha)\|_{H^1(\Omega_k)} + \|\lambda_k^{(0)} - \lambda_k(\alpha)\|_{L_2(S_k^-)} \right) + \|\chi_{12} v(\alpha)\|_{L_2(\Omega)} +$$
$$+ \|\chi_{12}(u_1(\alpha) - u_2(\alpha))\|_{L_2(\Omega)} \to 0 \text{ as } \alpha \to +0, \quad (64)$$

where $u_k(\alpha), \lambda_k(\alpha),\ k = 1, 2$ and $v(\alpha)$ is the solution of the optimal control problem (59) for $\alpha > 0$, while $u_k^{(0)}, \lambda_k^{(0)},\ k = 1, 2$ and $v^{(0)} \equiv 0$ denotes the solution of the exact controllability problem (56).
The convergence estimate (62) follows by the convergence of the iterative method (61), for suitable relaxation parameters γ_m ([1, 14, 22]). □

Remark 7.1 *The results proved in Proposition 7.1 remain valid for $\Omega \subset \mathbb{R}^3$ and for elliptic systems.*

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