A Mathematical Approach in the Design of Arterial Bypass Using Unsteady Stokes Equations

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Dedicated to David Gottlieb on the occasion of his 60^{th} birthday.
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In this paper we present an approach for the study of Aorto-Coronaric bypass anastomoses configurations using unsteady Stokes equations. The theory of optimal control based on adjoint formulation is applied in order to optimize the shape of the zone of the incoming branch of the bypass (the toe) into the coronary according to several optimality criteria.

KEY WORDS: Optimal Control; Shape Optimization; Small Perturbation Theory; Finite Elements; Unsteady Stokes Equations; Haemodynamics; Aorto-Coronaric Bypass Anastomoses; Design of Improved Medical Devices.

1 INTRODUCTION

In this paper we apply optimal control for the shape optimization of aorto-coronaric bypass anastomoses [8]. We analyze the "first correction" method which is derived by applying a perturbation method to the initial unsteady problem in a space-time domain $\Omega \times (0, T)$ with $\Omega \subset \mathbb{R}^2$. The boundary $\partial\Omega$ of Ω is parameterized by a suitable function f. Then we propose numerical methods for its solution.
Optimal control (Lions [4]) by perturbation theory (Van Dyke [17]) using adjoint equation techniques (Agoshkov [1] and Marchuk [6]) provides a tool

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for improving arterial bypass graft on the basis of a better understanding of the blood flow dynamics. In this paper we extend the approach and the results from [2] to the case where the *non-stationary* Stokes equations are used. An outline of this paper is as follows. In Sec.2 we introduce the problem statement, in Sec.3 we deal with the problem of perturbed functions in the generalized Stokes equations framework. In Sec.4 we introduce the shape optimization problem and its equations in the optimal control framework (Sec.5). In Sec.6 uniqueness and existence results are given, then in Sec.7 an iterative optimization algorithm is introduced. Sec.8 deals with a test problem and numerical results, finally some conclusions follow in Sec.9.

2 NOTATION AND PROBLEM STATEMENT

Let Ω be a bounded domain of $\mathbb{R}^2$ with boundary Γ, $\underline{x} := (x,y)$ is a point of $\overline{\Omega}$, $t \in [0,T], T < \infty$, is the time variable. For every scalar function ϕ and any vector function $\underline{v}$, whose components are u, v, we recall the definition of the following operators:

$$\nabla\phi = (\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}), \nabla\cdot\underline{v} := \mathcal{D}(\underline{v}) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y},$$

$$\nabla\times\underline{v} := \mathcal{R}\underline{v} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}.$$

In the sequel, vectors are indicated with underlined letters such as $\underline{v}$, while aggregation of vectors with scalars are indicated with underlined capital letters, such as $\underline{Q} = (\underline{v}, p)$. Consider an idealized, two-dimensional bypass bridge configuration in Fig.1 and the domain on Fig.2, where the dotted line represents the geometry of the complete anastomosis; Γ_{w_2} is the section of the original artery, Γ_{in} is the new anastomosis inflow after bypass surgery, Γ_{out} is the anastomosis outflow.

We consider the following boundary-value problem for the Stokes equations, used to model low Reynolds blood flow in this study: find $\underline{v}, p$ s.t.

$$\begin{cases} \underline{v}_t - \nu\Delta\underline{v} + \nabla p = \underline{F} \quad in\ \Omega\times(0,T), \\ \nabla\cdot\underline{v} = 0 \quad in\ \Omega\times(0,T), \\ \underline{v} = \underline{v}_{in}\ on\ \Gamma_{in},\ \underline{v} = 0\ on\ \Gamma_{w_1}\cup\Gamma_{w_3}\ \forall t\in(0,T) \\ -p\cdot\underline{n} + \nu\frac{\partial\underline{v}}{\partial n} = \underline{g}_{out} on\ \Gamma_{out}\cup\Gamma_{w_2}\ \forall t\in(0,T) \\ \underline{v} = \underline{v}^*\ at\ t=0\ in\ \Omega, \end{cases} \tag{1}$$

where $\underline{v}_t := \frac{\partial\underline{v}}{\partial t}$, $\underline{v}^*$ is a given vector function such that $\nabla\cdot\underline{v}^* = 0$ at $t = 0$ in Ω, $\underline{n} = (n_1, n_2)$ is the outward unit normal vector on Γ, $\underline{F} = \underline{F}(x,y,t)$, $\underline{v}_{in} = \underline{v}_{in}(x,y,t)$, $\underline{g}_{out} = \underline{g}_{out}(x,y,t)$ are given vector functions,

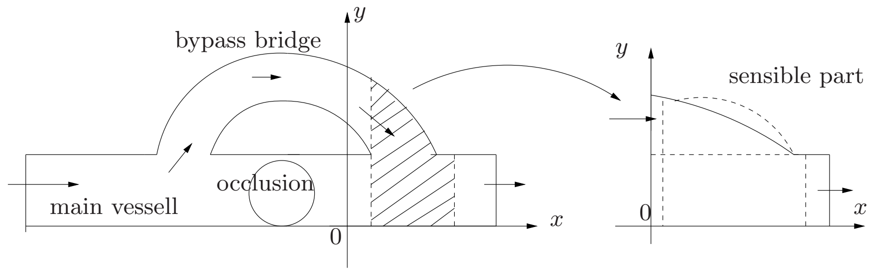

Figure 1: Idealized, 2-D bypass bridge configuration (left) and detail of the sensible part for the optimization process (right). The dotted curve represents the portion of the boundary that is subjected to change.

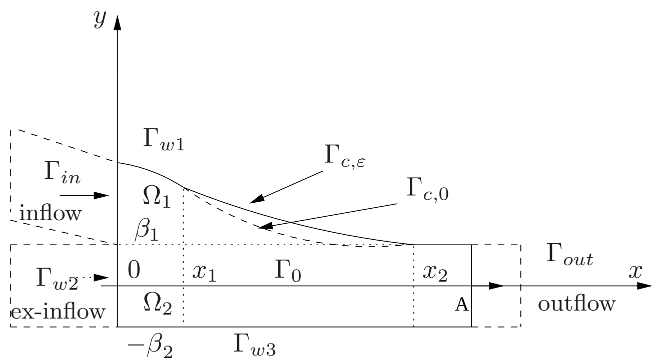

Figure 2: Main notation: $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2, \Gamma_w = \Gamma_{w_1} \cup \Gamma_{w_2} \cup \Gamma_{w_3}, \Gamma_0 = \partial\Omega_1 \cap \partial\Omega_2$

$\nu = const > 0$ and $v_f = \{\underline{v}_{in}\ on\ \Gamma_{in}; \underline{0}\ on\ \Gamma_{w_1} \cup \Gamma_{w_3}\}$. The subset $\Gamma_{c,\varepsilon}$ of Γ_{w_1} is parametrized by a function $f(x,\varepsilon)$ of $\underline{x} \in [x_1, x_2]$ and $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$, $\varepsilon_0 =$ *const* is a small parameter. More precisely we assume that $f(x,\varepsilon)$ can be developed as follows:

$$f(x,\varepsilon) = f_0(x) + \varepsilon f_1(x) + \varepsilon^2 f_2(x) + \ldots, \tag{2}$$

where $f_k \in W^{1,\infty}(x_1, x_2)$, for $k = 0$, and $f_k \in W_0^{1,\infty}(x_1, x_2)$, for $k \geq 1$, so that $f_k(x_1) = f_k(x_2) = 0, k \geq 1$. Here the function $f_0(x) > 0$ describes the original subset $\Gamma_{c,0}$ of the boundary of the "unperturbed domain", $\Gamma_{w0} \equiv \partial\Omega_0$ (see Fig. 3), while $f_k(x), k \geq 1$, could be unknown when dealing with control problem (see Sec.4).

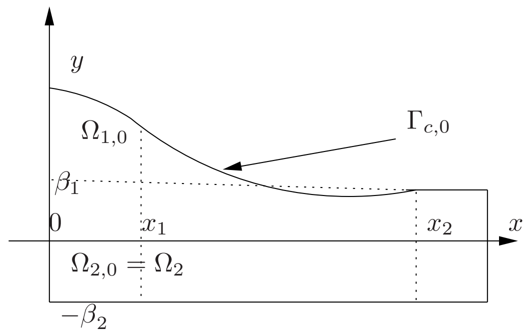

Figure 3: The original "unperturbed domain" Ω_0.

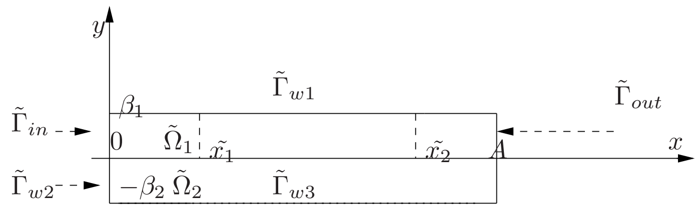

Figure 4: The "simple" domain $\tilde{\Omega}$.

We introduce the following real Hilbert spaces:

$$\mathbb{X} := \{\underline{\hat{v}} : \underline{\hat{v}} \in (H^1(\Omega))^2, \underline{\hat{v}} = 0 \ on \ \Gamma_{in} \cup \Gamma_{w_1} \cup \Gamma_{w_3}\},$$

$$H := L^2(\Omega \times (0,T)) \equiv H^*, \ Y := L^2(0,T;\mathbb{X}),$$

$$W := \{\underline{v} : \underline{v} \in Y, \underline{v}_t \in L^2(0,T;Y^*), \underline{v}(x,y,T) = 0\}.$$

The weak statement of Eq.(1) reads: find $\underline{v} \in L^2(0,T;(H^1(\Omega))^2)$, $p \in H$ s.t.

$$\begin{cases} a(\underline{v},\underline{\hat{v}}) - b(p,\underline{\hat{v}}) = G(\underline{\hat{v}}) \ \forall \underline{\hat{v}} \in W, \\ b(\hat{p},\underline{v}) = 0 \ \forall \hat{p} \in H, \\ \ \underline{v} = \underline{v}_f \ on \ \Gamma_{in} \cup \Gamma_{w_1} \cup \Gamma_{w_3}, \ \forall \ t \ \in \ (0,T) \end{cases} \tag{3}$$

where with $\underline{\hat{v}}$ we indicate test functions and:

$$a(\underline{v},\underline{\hat{v}}) = \int_0^T \int_\Omega \nu \nabla \underline{v} \cdot \nabla \underline{\hat{v}} d\Omega dt - \int_0^T \int_\Omega \underline{v} \cdot \underline{\hat{v}}_t d\Omega dt,$$

$$b(p,\underline{\hat{v}}) = \int_0^T \int_\Omega p \nabla \cdot \underline{\hat{v}} d\Omega dt,$$

$$G(\underline{\hat{v}}) = \int_0^T \int_\Omega \underline{F} \cdot \underline{\hat{v}} d\Omega dt + \int_0^T \int_{\Gamma_{out} \cup \Gamma_{w_2}} \underline{g}_{out} \cdot \underline{\hat{v}} d\Gamma dt + \int_\Omega \underline{v}^* \cdot \underline{\hat{v}}(x, y, 0) d\Omega.$$

The forms $a(.,.), b(.,.)$ and $G(.)$ depend on the parametrization f of $\Gamma_{c,\varepsilon}$, however this dependence will be understood for simplicity of notations.

3 THE PROBLEM FOR THE PERTURBED FUNCTIONS

Let us introduce the reference (simple-shaped) domains $\tilde{\Omega}_1 = \{0 < \tilde{x} < A, 0 < \tilde{y} < \beta_1 \equiv \beta\}$, $\tilde{\Omega}_2 = \{0 < \tilde{x} < A, -\beta_2 < \tilde{y} < 0\}$, and $\overline{\tilde{\Omega}} = \overline{\tilde{\Omega}}_1 \cup \overline{\tilde{\Omega}}_2$ (see Fig.4). Then we assume that $f(x, \varepsilon) > 0$ and consider the following variable transformation:

$$\mathrm{T}_f : \overline{\Omega}_1 \cup \overline{\Omega}_2 \to \overline{\tilde{\Omega}}, \ \underline{\tilde{x}} = \mathrm{T}_f(\underline{x});$$

T_f is the identity in Ω_2, while $\mathrm{T}_f(x, y) = (x, \frac{\beta}{f(x,\varepsilon)} y)$ in Ω_1. We set $\underline{\tilde{x}} = (\tilde{x}, \tilde{y})$ and define

$$\underline{\tilde{v}}(\underline{\tilde{x}}) := \underline{v} \circ \mathrm{T}_f^{-1}(\underline{\tilde{x}}) = \underline{v}(\tilde{x}, \tilde{y} f(\tilde{x}, \varepsilon)/\beta).$$

where $\underline{\tilde{v}} = (\tilde{u}, \tilde{v})$. Then,

$$dxdy = \frac{f(\tilde{x}, \varepsilon)}{\beta} d\tilde{x} d\tilde{y}$$

and the following relations hold (with $f_x := df/dx$):

$$\frac{\partial \phi}{\partial y}(\underline{\tilde{x}}) = \frac{\beta}{f(\tilde{x}, \varepsilon)} \frac{\partial \tilde{\phi}(\underline{\tilde{x}})}{\partial \tilde{y}}, \ \frac{\partial \phi}{\partial x}(\underline{\tilde{x}}) = \frac{\partial \tilde{\phi}(\underline{\tilde{x}})}{\partial \tilde{x}} - \tilde{y} \frac{f_x(\tilde{x}, \varepsilon)}{f(\tilde{x}, \varepsilon)} \frac{\partial \tilde{\phi}(\underline{\tilde{x}})}{\partial \tilde{y}}, \tag{4}$$

$$\begin{cases} \tilde{\mathcal{D}}(f)\underline{\tilde{v}}(\underline{\tilde{x}}) := ((\nabla \cdot \underline{v}) \circ \mathrm{T}_f^{-1})(\underline{\tilde{x}}) = \frac{\partial \tilde{u}}{\partial \tilde{x}} - \tilde{y} \frac{f_x(\tilde{x}, \epsilon)}{f(\tilde{x}, \varepsilon)} \frac{\partial \tilde{u}}{\partial \tilde{y}} + \frac{\beta}{f(\tilde{x}, \varepsilon)} \frac{\partial \tilde{v}}{\partial \tilde{y}}, \\ \tilde{\mathcal{R}}(f)\underline{\tilde{v}}(\underline{\tilde{x}}) := ((\nabla \times \underline{v}) \circ \mathrm{T}_f^{-1})(\underline{\tilde{x}}) = \frac{\partial \tilde{v}}{\partial \tilde{x}} - \tilde{y} \frac{f_x(\tilde{x}, \epsilon)}{f(\tilde{x}, \varepsilon)} \frac{\partial \tilde{v}}{\partial \tilde{y}} - \frac{\beta}{f(\tilde{x}, \varepsilon)} \frac{\partial \tilde{u}}{\partial \tilde{y}}. \end{cases} \tag{5}$$

Then in $\tilde{\Omega}$ we have:

$$\tilde{\mathcal{D}}(f)\underline{\tilde{v}} = m_2 \tilde{\nabla} \cdot \underline{\tilde{v}} + m_1 \tilde{\mathcal{D}}(f)\underline{\tilde{v}}, \ \tilde{\mathcal{R}}(f)\underline{\tilde{v}} = m_2 \tilde{\nabla} \times \underline{\tilde{v}} + m_1 \tilde{\mathcal{R}}(f)\underline{\tilde{v}},$$

where $\tilde{\nabla}\phi := \left(\frac{\partial \phi}{\partial \tilde{x}}, \frac{\partial \phi}{\partial \tilde{y}}\right)$, while m_s is the characteristic function of Ω_s $(s = 1, 2)$. To simplify the notations from now on we will set (unless otherwise specified):

$$\underline{\tilde{x}} = \underline{x}, \ \underline{\tilde{v}}(\tilde{x}, \tilde{y}, t) := \underline{v}(x, y, t), \ \tilde{u} = u, \ \tilde{v} = v, \ldots,$$

$$\tilde{\mathcal{D}} = \mathcal{D}, \tilde{\mathcal{R}} = \mathcal{R},\ \tilde{\Omega} \equiv \Omega,\ \tilde{\Gamma}_{w_k} \equiv \Gamma_{w_k}.$$

Then problem (3) in the new reference frame $\tilde{\Omega}$ reads as follows:

$$\begin{cases} a(f; \underline{v}, \hat{\underline{v}}) - b(f; p, \hat{\underline{v}}) = G(f; \hat{\underline{v}})\ \forall \hat{\underline{v}} \in W, \\ b(f; \hat{\underline{p}}, \underline{v}) = 0\ \forall \hat{p} \in H, \\ \underline{v} = \underline{v}_f \ on\ \Gamma_{in} \cup \Gamma_{w_1} \cup \Gamma_{w_3} \forall t \in (0, T). \end{cases} \tag{6}$$

We have emphasized the dependence of $a(f; .,.), b(f; .,.)$, and $G(f; .)$ on f. Precisely, upon writing Ω_1 instead of $\tilde{\Omega}_1$ and Ω_2 instead of $\tilde{\Omega}_2$ for simplicity of notation we have (unless otherwise specified, integration is carried out with respect to $dxdydt$):

$$a(f; \underline{v}, \hat{\underline{v}}) = a_1(f; \underline{v}, \hat{\underline{v}}) + a_2(\underline{v}, \hat{\underline{v}}),$$

$$a_1(f; \underline{v}, \hat{\underline{v}}) = \int_0^T \int_{\Omega_1} \frac{f\nu}{\beta}\Big(\Big(\frac{\partial \underline{v}}{\partial x} - \frac{y f_x}{f}\frac{\partial \underline{v}}{\partial y}\Big) \cdot \Big(\frac{\partial \hat{\underline{v}}}{\partial x} - \frac{y f_x}{f}\frac{\partial \hat{\underline{v}}}{\partial y}\Big) + \frac{\beta^2}{f^2}\frac{\partial \underline{v}}{\partial y} \cdot \frac{\partial \hat{\underline{v}}}{\partial y}\Big) -$$

$$- \int_0^T \int_{\Omega_1} \frac{f}{\beta} \underline{v} \cdot \hat{\underline{v}}_t,$$

$$a_2(\underline{v}, \hat{\underline{v}}) = \int_0^T \int_{\Omega_2} \nu\Big(\frac{\partial \underline{v}}{\partial x} \cdot \frac{\partial \hat{\underline{v}}}{\partial x} + \frac{\partial \underline{v}}{\partial y} \cdot \frac{\partial \hat{\underline{v}}}{\partial y}\Big) - \int_0^T \int_{\Omega_2} \underline{v} \cdot \hat{\underline{v}}_t,$$

$$b(f; p, \hat{\underline{v}}) = b_1(f; p, \hat{\underline{v}}) + b_2(p, \hat{\underline{v}}),$$

$$b_1(f; p, \hat{\underline{v}}) = \int_0^T \int_{\Omega_1} \frac{f}{\beta} p \mathcal{D}(f) \hat{\underline{v}},\quad b_2(p, \hat{\underline{v}}) = \int_0^T \int_{\Omega_2} p \nabla \cdot \hat{\underline{v}},$$

$$G(f; \hat{\underline{v}}) = G_1(f; \hat{\underline{v}}) + G_2(\hat{\underline{v}}),$$

$$G_1(f; \hat{\underline{v}}) = \int_0^T \int_{\Omega_1} \frac{f}{\beta} \underline{F} \cdot \hat{\underline{v}} + \int_0^T \int_{(\Gamma_{out} \cup \Gamma_{w2}) \cap \partial\Omega_1} \underline{g}_{out} \cdot \hat{\underline{v}} d\Gamma dt +$$

$$+ \int_{\Omega_1} \frac{f}{\beta} \underline{v}^*(x, y) \cdot \hat{\underline{v}}(x, y, 0) dxdy,$$

$$G_2(\hat{\underline{v}}) = \int_0^T \int_{\Omega_2} \underline{F} \cdot \hat{\underline{v}} + \int_0^T \int_{(\Gamma_{out} \cup \Gamma_{w2}) \cap \partial\Omega_2} \underline{g}_{out} \cdot \hat{\underline{v}} d\Gamma dt +$$

$$+ \int_{\Omega_2} \underline{v}^*(x, y) \cdot \hat{\underline{v}}(x, y, 0) dxdy.$$

Note that in the sequel the test functions $\hat{\underline{v}},\ \hat{p}$ in Eq.(6) can be assumed to be independent of ε.
Assume that problem in Eq.(6) has a solution $\underline{v}, p$ that is infinitely differentiable with respect to ε:

$$\begin{cases} \underline{v} = \underline{v}_0 + \varepsilon \underline{v}_1 + \varepsilon^2 \underline{v}_2 + \dots \\ p = p_0 + \varepsilon p_1 + \varepsilon^2 p_2 + \dots \end{cases} \tag{7}$$

where $p_k \in H, \underline{v}_k \in Y,\ k \geq 1$. Using Eqs.(2)-(7) and the small perturbation technique we can deduce the equations satisfied by $\underline{v}_k, p_k,\ k \geq 0$. In particular, for $k = 0$, $\underline{v}_0$ and p_0 satisfy

$$\begin{cases} a(f_0; \underline{v}_0, \hat{\underline{v}}) - b(f_0; p_0, \hat{\underline{v}}) = G(f_0; \hat{\underline{v}}) \ \ \forall \hat{\underline{v}} \in W, \\ b(f_0; \hat{\underline{p}}, \underline{v}_0) = 0 \ \ \forall \hat{p} \in H, \\ \ \underline{v}_0 = \underline{v}_f \ \mathit{on} \ \Gamma_{in} \cup \Gamma_{w_1} \cup \Gamma_{w_3} \ \forall t \in (0,T). \end{cases} \tag{8}$$

Correspondingly we define:

$$\mathcal{R}_{obs,0} := \mathcal{R}(f_0)\underline{v}_0. \tag{9}$$

We introduce additionally functional spaces $\mathbb{H}^p$ and $\mathbb{H}_f$ for p and $\{f_k\}$, respectively s.t.:

$$\mathbb{H}^p \subseteq H \subseteq \mathbb{H}^{p*},\ \mathbb{H}_f \subseteq L^2(x_1, x_2) \subseteq \mathbb{H}_f^*,$$

$$\mathbb{W} := W \times \mathbb{H}^p \subseteq \mathbb{H}_0 := L^2(\Omega \times (0,T))^2 \times \mathbb{L}^2(\Omega \times (0,T)) \subseteq \mathbb{W}^*,$$

Then we set:

$$\mathbb{Y} := Y \times \mathbb{H}^p \subseteq \mathbb{H}_0 \subseteq \mathbb{Y}^*,$$

Then for $k = 1$ the functions $\underline{v_1}, p_1$, considered as the components of the vector-function $\underline{\Phi_1} := (\underline{v_1}, p_1) \in \mathbb{Y},\ f_1 \in \mathbb{H}_f$, satisfy the equation:

$$\mathcal{L}(\underline{\Phi_1}, \hat{\underline{\Phi}}) = B(f_1, \hat{\underline{\Phi}})\ \forall \hat{\underline{\Phi}} := (\hat{\underline{v}}, \hat{p}) \in \mathbb{W}, \tag{10}$$

where

$$\mathcal{L}(\underline{\Phi_1}, \hat{\underline{\Phi}}) := a_0(f_0; \underline{v_1}, \hat{\underline{v}}) - b_0(f_0; p_1, \hat{\underline{v}}) + b_0(f_0; \hat{p}, \underline{v_1}),$$

$$B(f_1, \hat{\underline{\Phi}}) := b_f(f_1; p_0, \hat{\underline{v}}) + G_1(f_1; \hat{\underline{v}}) - a_f(f_1; \underline{v}_0, \hat{\underline{v}}) - b_f(f_1; \hat{p}, \underline{v}_0),$$

$$b_f(f_1; p_0, \hat{\underline{v}}) := \frac{\partial}{\partial \varepsilon} b(f; p_0, \hat{\underline{v}})|_{\varepsilon=0} = \int_0^T \int_{\Omega_1} \frac{f_1}{\beta} p_0 \mathcal{D}(f_0)\hat{\underline{v}} +$$

$$+ \int_0^T \int_{\Omega_1} \frac{f_0}{\beta} p_0 \mathcal{D}_f(f_1, \hat{\underline{v}}),$$

$$\mathcal{D}_f(f_1, \hat{\underline{v}}) := \frac{\partial}{\partial \varepsilon} \mathcal{D}(f)\hat{\underline{v}}|_{\varepsilon=0} = -\big[y\big(\frac{f_{1,x} f_0 - f_{0,x} f_1}{f_0^2}\big)\frac{\partial \hat{u}}{\partial y} + \frac{\beta f_1}{f_0^2}\frac{\partial \hat{v}}{\partial y}\big]$$

$$\mathcal{D}_f(f_1, \underline{v}_0) := \frac{\partial}{\partial \varepsilon} \mathcal{D}(f)\underline{v}_0|_{\varepsilon=0} (:= \mathcal{D}_f f_1 \ \mathit{in\ the\ sequel}),$$

$$G_1(f_1; \hat{\underline{v}}) := \frac{\partial}{\partial \varepsilon} G(f; \hat{\underline{v}})|_{\varepsilon=0} = \int_0^T \int_{\Omega_1} \frac{f_1}{\beta} F \cdot \hat{\underline{v}} + \int_{\Omega_1} \frac{f_1}{\beta} \underline{v}_0(x,y) \cdot \hat{\underline{v}}(x,y,0) dx dy,$$

$$a_f(f_1; \underline{v}_0, \hat{\underline{v}}) := \frac{\partial}{\partial \varepsilon} a(f; \underline{v}_0, \hat{\underline{v}})|_{\varepsilon=0} =$$

$$= \int_0^T \int_{\Omega_1} \frac{f_1\nu}{\beta}((\frac{\partial \underline{v}_0}{\partial x} - \frac{y f_{0,x}}{f_0}\frac{\partial \underline{v}_0}{\partial y}) \cdot (\frac{\partial \hat{\underline{v}}}{\partial x} - \frac{y f_{0,x}}{f_0}\frac{\partial \hat{\underline{v}}}{\partial y}) + \frac{\beta^2}{f_0^2}\frac{\partial \underline{v}_0}{\partial y} \cdot \frac{\partial \hat{\underline{v}}}{\partial y}) -$$

$$- \int_0^T \int_{\Omega_1} \frac{f_0\nu}{\beta} y \frac{(f_{1,x}f_0 - f_{0,x}f_1)}{f_0^2}(\frac{\partial \underline{v}_0}{\partial y} \cdot (\frac{\partial \hat{\underline{v}}}{\partial x} - \frac{y f_{0,x}}{f_0}\frac{\partial \hat{\underline{v}}}{\partial y}) + (\frac{\partial \underline{v}_0}{\partial x} - \frac{y f_{0,x}}{f_0}\frac{\partial \underline{v}_0}{\partial y}) \cdot \frac{\partial \hat{\underline{v}}}{\partial y})$$

$$- \int_0^T \int_{\Omega_1} \frac{f_0\nu}{\beta}(\frac{2\beta^2 f_1}{f_0^3})\frac{\partial \underline{v}_0}{\partial y} \cdot \frac{\partial \hat{\underline{v}}}{\partial y} - \int_0^T \int_{\Omega_1} \frac{f_1}{\beta}\underline{v}_0 \cdot \hat{\underline{v}}_t.$$

This is a weak statement for the non-stationary Stokes problem. By a similar technique we can derive the equations for $\underline{v}_k,\ p_k$ with $k \geq 2$. However we will not carry on this development any further in this work. In the sequel we assume that the non-stationary Stokes problem in Eq.(10) has a unique solution for any given $\underline{v}_0,\ p_0$ (the solution in the unperturbed domain Ω_0) and for each $f_1 \in \mathbb{H}_f$.

4 THE SHAPE OPTIMIZATION PROBLEM

Suppose now that the function f_1 in Eq.(10) is unknown and so are $\underline{v}_1,\ p_1$. To complete problem (10) we will have either to provide some additional equations or to require that f_1 be determined by minimizing a suitable cost functional.
Problem (10) can be supplemented by the additional equation:

$$\mathcal{C}(f, \underline{v}, p) = 0 \tag{11}$$

where $\mathcal{C}$ is an operator (linear or nonlinear) defined on $H_0^1(x_1, x_2) \times Y \times \mathbb{H}^p$. (We consider now $f \in H_0^1$ for convenience). We assume $\mathcal{C}$ to depend smoothly on its variables $f, \underline{v}, p$. Using the representations (2) and (7) we derive from (11) the following equation:

$$\mathcal{C}(f, \underline{v}, p) = \mathcal{C}(f_0, \underline{v}_0, p_0) + \varepsilon \mathcal{C}_1(f_1, \underline{v}_1, p_1) + \mathcal{O}(\varepsilon^2) = 0, \quad \forall \varepsilon \in [-\varepsilon_0, \varepsilon_0] \tag{12}$$

where

$$\mathcal{C}_1(f_1, \underline{v}_1, p_1) := \frac{\partial \mathcal{C}}{\partial \varepsilon}(f, \underline{v}, p)|_{\varepsilon=0}. \tag{13}$$

If we assume that the data of our problems are such that $\mathcal{C}(f_0, \underline{v}_0, p_0) = 0$, then we can replace (12) by the approximate equation

$$\mathcal{C}_1(f_1, \underline{v}_1, p_1) = 0 \tag{14}$$

and use it to complete (10). An alternative approach would consist in replacing the *exact controllability equation* (14) by the following equivalent *minimization problem*:

$$\inf_{f_1} \int_0^T \int_\Omega \frac{f_0}{\beta}|\mathcal{C}_1(f_1, \underline{v}_1, p_1)|^2 dxdydt, \tag{15}$$

where we assume that $\mathcal{C}_1$ has image in $\mathbb{H}^p$. In the next sections we apply the approach described above for the completion of (10) and we will use the following special choice of (11):

$$\mathcal{C}(f,\underline{v}) := ((\nabla \times \underline{v}) \circ \mathrm{T}_f^{-1})(x,y,t) - \mathcal{R}_{obs,\varepsilon}(x,y,t) \ in \ \Omega_{wd} \subseteq \Omega \ \forall t, \quad (16)$$

where Ω_{wd} is a suitable subset of Ω in which we want our additional equation (or our "control") to take place. Moreover

$$\mathcal{R}_{obs,\varepsilon} = \mathcal{R}_{obs,0} + \varepsilon \mathcal{R}_{obs,1} + \varepsilon^2 \mathcal{R}_{obs,2} + \dots \ , \ \mathcal{R}_{obs,0} := ((\nabla \times \underline{v}_0) \circ \mathrm{T}_{f_0}^{-1}). \ (17)$$

Then we have: $\mathcal{C}(f_0, \underline{v}_0) = 0$, while the equation (14) reads:

$$\mathcal{C}(f_1, \underline{v}_1) = \mathcal{R}(f_0)\underline{v}_1 + m_1 \mathcal{R}_f f_1 - \mathcal{R}_{obs,1} = 0 \ in \ \Omega_{wd}, \ \forall t, \quad (18)$$

where

$$\mathcal{R}(f_0)\underline{v}_1 = (\nabla \times \underline{v}_1) \circ \mathrm{T}_{f_0}^{-1}(x,y) = \frac{\partial v_1}{\partial x} - \frac{y f_{0,x}}{f_0}\frac{\partial v_1}{\partial y} - \frac{\beta}{f_0}\frac{\partial u_1}{\partial y},$$

$$\mathcal{R}_f f_1 := \mathcal{R}_f(f_1, \underline{v}_0) = -y\frac{(f_{1,x} f_0 - f_{0,x} f_1)}{f_0^2}\frac{\partial v_0}{\partial y} + \frac{\beta f_1}{f_0^2}\frac{\partial u_0}{\partial y}.$$

Therefore we have the problem: find $\underline{\Phi_1} = (\underline{v_1}, p_1) \in \mathbb{Y}, \ f_1 \in H_0^1(x_1, x_2)$ s.t.

$$\begin{cases} \mathcal{L}(\underline{\Phi_1}, \hat{\underline{\Phi}}) = B(f_1, \hat{\underline{\Phi}}) \ \forall \hat{\underline{\Phi}} = (\hat{\underline{v}}, \hat{p}) \in \mathbb{W}, \\ \mathcal{R}(f_0)\underline{v}_1 + m_1 \mathcal{R}_f f_1 - \mathcal{R}_{obs,1} = 0 \ in \ \Omega_{wd} \ \forall t, \end{cases} \quad (19)$$

where $\mathcal{R}_{obs,1}$ is a given function. Problem (19) is an "exact controllability problem". These problems have solutions in some particular cases only. For this reason we replace (19) by the following generalized optimal control problem: find $\underline{\Phi_1} = (\underline{v_1}, p_1) \in \mathbb{Y}, \ f_1 \in H_0^1(x_1, x_2)$ s.t.

$$\begin{cases} \mathcal{L}(\underline{\Phi_1}, \hat{\underline{\Phi}}) = B(f_1, \hat{\underline{\Phi}}) \ \forall \hat{\underline{\Phi}} = (\hat{\underline{v}}, \hat{p}) \in \mathbb{W}, \\ \inf_{f_1} = \frac{\alpha}{2}||f_1||^2_{H_0^1(x_1,x_2)} + J(f_1, \underline{v}_1, p_1), \end{cases} \quad (20)$$

where

$$J(f_1, \underline{v}_1, p_1) = \gamma_1 J_1(f_1, \underline{v}_1) + \gamma_2 J_2(f_1, \underline{v}_1, p_1) + \gamma_3 J_3(f_1, \underline{v}_1, p_1),$$

$$J_1(f_1, \underline{v}_1) = \frac{1}{2}\int_0^T \int_\Omega m_{wd} \frac{f_0}{\beta}(\mathcal{R}(f_0)\underline{v}_1 + m_1 \mathcal{R}_f f_1 - \mathcal{R}_{obs,1})^2,$$

$\alpha = const \geq 0$ is a small regularization parameter, $\gamma_1 > 0$ is a weight, m_{wd} is the characteristic function of Ω_{wd}. This functional allows the control of a term related with the vorticity which is a relevant clinical index (see, for example, [8]).

Note that the second equation from (19) is considered in (20) in the least square sense; then (20) for $\alpha = 0, \gamma_2 = \gamma_3 = 0$ provides the weak statement of problem (19). Otherwise the solution $v_1 = v_1(\alpha)$, $p_1 = p_1(\alpha)$, $f_1 = f_1(\alpha)$ of (20) represents an approximate (regularized) solution of (19). In the sequel γ_2, γ_3 are non-negative constant weight coefficients, while $J_2(f_1, \underline{v}_1, p_1)$ and $J_3(f_1, \underline{v}_1, p_1)$ are additional functionals that are assumed to be quadratic.
An example of $J_2(f_1, \underline{v}_1, p_1)$ follows:

$$J_2(f_1, \underline{v}_1, p_1) = J_2(\underline{v}_1, p_1) := \tag{21}$$

$$= \frac{1}{2}\Big(||p_1 - p_{out,1}||^2_{L^2(\Gamma_{out}\times(0,T))} + ||\underline{v}_1 - \underline{v}_{out,1}||^2_{L^2(\Gamma_{out}\times(0,T))^2} \Big)$$

The functional J_3 is introduced in order to enhance the smoothness in time of $\underline{v}_1, \ldots, p_1$. There we will take:

$$J_3(f_1, \underline{v}_1, p_1) = J_3(\tau; p_1) := \frac{1}{2}\|\mathcal{J}(p_1 - p_{out,1})\|^2_{\mathbb{L}^2(\Gamma_{out}\times(0,T))},$$

where

$$\mathcal{J}p := \frac{p(x,y,t) - p(x,y,t-\tau)}{\tau},$$

$\tau \geq 0$ is a parameter and we assume the functions $p_1, p_{out,1}$ to be extended by parity to the negative values of t (i.e. $p(x,y,t) := p(x,y,-t)$ as $t < 0, (x,y) \in \Omega$, etc). If $p_1, p_{out,1} \in H^1(0,T; L^2(\Omega))$ and $\tau \to 0$ then $J_3 \to \frac{1}{2}\|p_{1,t} - (p_{out,1})_t\|^2_{\mathbb{L}^2(\Gamma_{out}\times(0,T))}$, i.e. by means of J_3 we impose a regularity restriction to p_1. If $\tau \to \infty$, then $J_3 \to 0$ and no additional restriction holds on p_1. If $0 < \tau < \infty$ then the introduction of J_3 can be regarded as a tool that yields a regularization condition for p_1 on $\Gamma_{out} \times (0,T)$. (Of course there are other ways to introduce similar regularity restrictions).

5 THE VARIATIONAL EQUATIONS

When considering (20) we can still consider the simple domain Ω of Fig.4. An alternative possibility (that we are going to follow) consists of using the new variable transformation

$$T^{-1}_{f_0}(\underline{\tilde{x}}) = \underline{x},\ \underline{\tilde{x}} \in \Omega,\ \underline{x} \in \Omega_0, \tag{22}$$

which is the identity in $\tilde{\Omega}_2$, while $T^{-1}_{f_0}(\tilde{x}, \tilde{y}) = (\tilde{x}, \frac{f_0(\tilde{x})}{\beta}\tilde{y})$ in $\tilde{\Omega}_1$, then working in the "unperturbed" domain Ω_0 (see Fig.3) where the expressions for the bilinear forms in (20) become simpler. Indeed in $\Omega_0 \times (0,T)$ problem

(20) can be reformulated as follows: find $\underline{\Phi} = (\underline{v}, p) := \underline{\Phi_1} = (\underline{v_1}, p_1) \in \mathbb{Y}$, $f := f_1 \in \mathbb{H}_f$[5] , s.t

$$\left\{ \begin{array}{l} \mathcal{L}(\underline{\Phi}, \underline{\hat{\Phi}}) = B(f, \underline{\hat{\Phi}}) \ \forall \underline{\hat{\Phi}} := (\underline{\hat{v}}, \hat{p}) \in \mathbb{W}, \\ \inf_{f \in \mathbb{H}_f} = \frac{\alpha}{2} ||f||^2_{H^1_0(x_1,x_2)} + J(f, \underline{\Phi}), \end{array} \right. \tag{23}$$

where

$$\mathcal{L}(\underline{\Phi}, \underline{\hat{\Phi}}) = a_0(\underline{v}, \underline{\hat{v}}) - b_0(p, \underline{\hat{v}}) + b_0(\hat{p}, \underline{v}),$$

$$B(f, \underline{\hat{\Phi}}) := b_f(f, p_0, \underline{\hat{v}}) + G_1(f, \underline{\hat{v}}) - a_f(f, \underline{v}_0, \underline{\hat{v}}) - b_f(f, \hat{p}, \underline{v}_0),$$

$$a_0(\underline{v}, \underline{\hat{v}}) = \int_0^T \int_{\Omega_0} \nu\Big(\frac{\partial \underline{v}}{\partial x} \cdot \frac{\partial \underline{\hat{v}}}{\partial x} + \frac{\partial \underline{v}}{\partial y} \cdot \frac{\partial \underline{\hat{v}}}{\partial y}\Big) - \int_0^T \int_{\Omega_0} \underline{v} \cdot \underline{\hat{v}}_t,$$

$$b_0(p, \underline{\hat{v}}) = \int_0^T \int_{\Omega_0} p \nabla \cdot \underline{\hat{v}},$$

$$b_f(f, p_0, \underline{\hat{v}}) = \int_0^T \int_{\Omega_{0,1}} p_0 \mathcal{D}_f(f, \underline{\hat{v}}) + \int_0^T \int_{\Omega_{0,1}} \frac{f}{f_0} p_0 \nabla \cdot \underline{\hat{v}},$$

$$\mathcal{D}_f(f, \underline{\hat{v}}) = -\big[y\Big(\frac{f_x f_0 - f_{0,x} f}{f_0^2}\Big)\frac{\partial \hat{u}}{\partial y} + \frac{f}{f_0}\frac{\partial \hat{v}}{\partial y}\big],$$

$$\mathcal{D}_f(f, v_0) := \mathcal{D}_f f,$$

$$G_1(f; \underline{\hat{v}}) = \int_0^T \int_{\Omega_{0,1}} \frac{f}{f_0} \underline{F} \cdot \underline{\hat{v}} + \int_{\Omega_{0,1}} \frac{f}{f_0} \underline{v}_0 \cdot \underline{\hat{v}}(x, y, 0) dx dy,$$

$$a_f(f; \underline{v}_0, \underline{\hat{v}}) = \int_0^T \int_{\Omega_{0,1}} \frac{f\nu}{f_0} \nabla \underline{v}_0 \cdot \nabla \underline{\hat{v}} - \int_0^T \int_{\Omega_{0,1}} \nu y \frac{(f_x f_0 - f_{0,x} f)}{f_0^2} \Big(\frac{\partial \underline{v}_0}{\partial y} \cdot \frac{\partial \underline{\hat{v}}}{\partial x} + \frac{\partial \underline{v}_0}{\partial x} \cdot \frac{\partial \underline{\hat{v}}}{\partial y}\Big) +$$

$$- \int_0^T \int_{\Omega_{0,1}} \frac{2f\nu}{f_0} \frac{\partial \underline{v}_0}{\partial y} \cdot \frac{\partial \underline{\hat{v}}}{\partial y},$$

$$J(f, \underline{v}, p) = \gamma_1 J_1(f, \underline{v}) + \gamma_2 J_2(f, \underline{v}, p) + \gamma_3 J_3(\tau; p),$$

$$J_1(f, \underline{v}) = \frac{1}{2} \int_0^T \int_{\Omega_0} m_{wd} |\nabla \times \underline{v} + m_1 \mathcal{R}_f f - \mathcal{R}_{obs,1}|^2, \tag{24}$$

$$\mathcal{R}_f f := \mathcal{R}_f(f, \underline{v}_0) = -y \frac{(f_x f_0 - f_{0,x} f)}{f_0^2} \frac{\partial v_0}{\partial y} + \frac{f}{f_0} \frac{\partial u_0}{\partial y}$$

$J_2(f, \underline{v}, p)$ and $J_3(\tau, p)$ are defined similarly. Let us derive the operator form of problem (23). Should $\underline{\Phi}$ be a solution of (23), then

$$\alpha(f, \hat{f})_{\mathbb{H}_f} + \langle J'_\Phi(f, \underline{\Phi}), \underline{\Phi}_{\hat{f}} \rangle + \langle J'_f(f, \underline{\Phi}), \hat{f} \rangle = 0, \tag{25}$$

[5] From now on we denote $\underline{v}_1 = \underline{v}$, $p_1 = p$, $f_1 = f$ however we should keep in mind that now $\underline{v}, p, f$ represent the "first corrections" of $\underline{v}_0$, p_0, f_0 on the unperturbed domain.

for any $\hat{f} \in \mathbb{H}_f$ ($\hat{f}$ is the independent variation), where $\underline{\Phi}_{\hat{f}} \in \mathbb{W}$ satisfies the following equation:

$$\mathcal{L}(\underline{\Phi}_{\hat{f}}, \underline{\hat{\Phi}}) = B(\hat{f}, \underline{\hat{\Phi}}) \ \forall \underline{\hat{\Phi}} \in \mathbb{W}. \tag{26}$$

In (25), $J'_\Phi = \frac{\partial J}{\partial \underline{\Phi}}$ and $J'_f = \frac{\partial J}{\partial f}$ are partial derivatives of J, while $\langle Q, \underline{\Phi} \rangle$ stands for ${}_{\mathbb{W}}\langle Q, \underline{\Phi} \rangle_{\mathbb{W}^*}$ the duality between $\mathbb{W}$ and $\mathbb{W}^*$ and $\langle g, f \rangle$ for the duality ${}_{\mathbb{H}_f}\langle g, f \rangle_{\mathbb{H}_f^*}$ between $\mathbb{H}_f$ and $\mathbb{H}_f^*$. Then we can write for (23) the system of "optimality conditions":

$$\begin{cases} \mathcal{L}(\underline{\Phi}, \underline{\hat{\Phi}}) = B(f, \underline{\hat{\Phi}}) \ \forall \underline{\hat{\Phi}} \in \mathbb{W}, \\ \alpha(f, \hat{f})_{\mathbb{H}^f} + \langle J'_\Phi(f, \underline{\Phi}), \underline{\Phi}_{\hat{f}} \rangle + \langle J'_f(f, \underline{\Phi}), \hat{f} \rangle = 0 \ \forall \hat{f} \in \mathbb{H}_f. \end{cases} \tag{27}$$

The element $\underline{\Phi}_{\hat{f}}$ can be eliminated from (27) by introducing the adjoint problem: find $\underline{Q} := (\underline{q}, \sigma) \in \mathbb{W}$ s.t.

$$\mathcal{L}^*(\underline{Q}, \underline{\hat{W}}) := \mathcal{L}(\underline{\hat{W}}, \underline{Q}) = \langle J'_\Phi(f, \underline{\Phi}), \underline{\hat{W}} \rangle \ \forall \underline{\hat{W}} \in \mathbb{Y}. \tag{28}$$

Since $\underline{\Phi}_{\hat{f}} \in \mathbb{Y}$ we can choose $\underline{\hat{W}} = \underline{\Phi}_{\hat{f}}$ in (28), yielding

$$\langle J'_\Phi(f, \underline{\Phi}), \underline{\Phi}_{\hat{f}}) \rangle = \mathcal{L}(\underline{\Phi}_{\hat{f}}, \underline{Q}) = B(\hat{f}, \underline{Q}) \tag{29}$$

and the system of variational equations (27) reads now as follows:

$$\begin{cases} \mathcal{L}(\underline{\Phi}, \underline{\hat{\Phi}}) = B(f, \underline{\hat{\Phi}}) \ \forall \underline{\hat{\Phi}} \in \mathbb{W}, \\ \mathcal{L}^*(\underline{Q}, \underline{\hat{W}}) = \langle J'_\Phi(f, \underline{\Phi}), \underline{\hat{W}} \rangle \ \forall \underline{\hat{W}} \in \mathbb{Y}, \\ \alpha(f, \hat{f})_{\mathbb{H}_f} + B(\hat{f}, \underline{Q}) + \langle J'_f(f, \underline{\Phi}), \hat{f} \rangle = 0 \ \forall \hat{f} \in \mathbb{H}_f. \end{cases} \tag{30}$$

The first equation is the state equation. Let us define the following operators (see [5], [4], [1]):

$$L : \mathbb{Y} \to \mathbb{W}^*, \ (L\underline{\Phi}, \underline{\hat{\Phi}})_{\mathbb{H}_0} := \mathcal{L}(\underline{\Phi}, \underline{\hat{\Phi}}),$$
$$L^* : \mathbb{W} \to \mathbb{Y}^*, \ (\underline{\hat{W}}, L^*\underline{Q})_{\mathbb{H}_0} = (L\underline{\hat{W}}, \underline{Q})_{\mathbb{H}_0},$$
$$B : \mathbb{H}_f \to \mathbb{W}^*, \ (Bf, \underline{\Phi})_{\mathbb{H}_0} = B(f, \underline{\Phi})$$
$$\Lambda_w : \mathbb{Y}^* \to \mathbb{Y}^*, \ (\Lambda_w J_\Phi(f, \underline{\Phi}), \underline{\hat{W}})_{\mathbb{H}_0} := \langle J'_\Phi(f, \underline{\Phi}), \underline{\hat{W}} \rangle,$$
$$\Lambda_f : \mathbb{H}_f^* \to \mathbb{H}_f^*, \ (\Lambda_f J_f(f, \underline{\Phi}), \hat{f})_{\mathbb{L}^2(x_1, x_2)} := \langle J'_f(f, \underline{\Phi}), \hat{f} \rangle.$$

Now the system (30) can be written in operator form as follows:

$$\begin{cases} L\underline{\Phi} = Bf \quad (in\ \mathbb{W}^*), \\ L^*\underline{Q} = \Lambda_w J_\Phi(f, \underline{\Phi}) \quad (in\ \mathbb{Y}^*), \\ \alpha\Lambda_c f + B^*\underline{Q} + \Lambda_f J_f(f, \underline{\Phi}) = 0 \quad (in\ (\mathbb{H}_f)^*), \end{cases} \tag{31}$$

where Λ_c is the extension to $\mathbb{H}_f$ of the operator:

$$\Lambda_{c,0} f := -f_{xx} + f$$

whose domain is $D(\Lambda_{c,0}) = H^2 \cap \mathbb{H}_f$.

6 UNIQUENESS AND EXISTENCE RESULTS

We analyze the particular cases where the cost functional J is chosen as outlined in Sec.4. Let J be one of the functionals J_2, J_3 of Section 4. Then

$$J(f,\underline{\Phi}) = J(f,\underline{v},p) = \frac{\gamma_1}{2}\int_0^T\int_{\Omega_0} m_{wd}|\nabla\times\underline{v} + m_1\mathcal{R}_f f - \mathcal{R}_{obs,1}|^2 d\Omega dt+ \tag{32}$$

$$+\frac{\gamma_2}{2}\int_0^T\int_{\Gamma_{out}}\Big(|p-p_{out}|^2+|\underline{v}-\underline{v}_{out}|^2\Big)d\Gamma dt+\frac{\gamma_3}{2}\int_0^T\int_{\Gamma_{out}}|\mathcal{J}(p-p_{out})|^2 d\Gamma dt$$

We assume that $\Omega_{wd} = \Omega_0$ and we define the spaces:

$$\mathbb{X} := \{\underline{v} : \underline{v} \in (H^2(\Omega))^2, \underline{v} = 0 \ on \ \Gamma_{in}\cup\Gamma_{w_1}\cup\Gamma_{w_3}\},$$

$$\mathbb{H}^p := \mathbb{L}^2(0,T;H^1(\Omega_0)),\ \mathbb{H}_f := H^2(x_1,x_2)\cap H_0^1(x_1,x_2).$$

Here we pretend that the velocity be in H^2 in order to use the uniqueness continuation theorem. The derivatives $J'_\Phi(f,\underline{\Phi})$ and $J'_f(f,\underline{\Phi})$ become

$$\langle J'_\Phi(f,\underline{\Phi}),\hat{\underline{\Phi}}\rangle = \gamma_1\int_0^T\int_{\Omega_0} m_{wd}(\nabla\times\underline{v} + m_1\mathcal{R}_f f - \mathcal{R}_{obs,1})\cdot(\nabla\times\hat{\underline{v}})d\Omega dt+$$

$$+\gamma_2\int_0^T\int_{\Gamma_{out}}(p-p_{out})\hat{p}d\Gamma dt + \gamma_2\int_0^T\int_{\Gamma_{out}}(\underline{v}-\underline{v}_{out})\cdot\hat{\underline{v}}d\Gamma dt+$$

$$+\gamma_3\int_0^T\int_{\Gamma_{out}}\mathcal{J}(p-p_{out})\cdot\mathcal{J}\hat{p}d\Gamma dt,$$

$$\langle J'_f(f,\underline{\Phi}),\hat{f}\rangle = \gamma_1\int_0^T\int_{\Omega_0} m_{wd}(\nabla\times\underline{v} + m_1\mathcal{R}_f f - \mathcal{R}_{obs,1})\mathcal{R}_f\hat{f}d\Omega dt,$$

$$\forall\ \hat{\underline{\Phi}} = (\hat{\underline{v}},\hat{p})\ and\ \forall\ \hat{f}.$$

The system of variational equations (27) becomes: find $\underline{\Phi_f} = (\underline{v}_f, p_f) \in Y\times\mathbb{H}^p$ s.t.

$$\begin{cases} \mathcal{L}(\underline{\Phi_f},\hat{\underline{\Phi}}) = B(f,\hat{\underline{\Phi}})\ \forall\hat{\underline{\Phi}}\in W\times\mathbb{H}^p, \\ \alpha(f,\hat{f})_{\mathbb{H}_f} + \gamma_1\int_0^T\int_{\Omega_0} m_{wd}(\nabla\times\underline{v}_f + m_1\mathcal{R}_f f - \mathcal{R}_{obs,1})\cdot(\nabla\times\underline{v}_{\hat{f}}+ \\ +m_1\mathcal{R}_f\hat{f})d\Omega dt + \gamma_2\int_0^T\int_{\Gamma_{out}}((p_f-p_{out})p_{\hat{f}} + (\underline{v}_f-\underline{v}_{out})\cdot\underline{v}_{\hat{f}})d\Gamma dt \\ +\gamma_3\int_0^T\int_{\Gamma_{out}}\mathcal{J}(p_f-p_{out})\mathcal{J}p_{\hat{f}}d\Gamma dt = 0\ \forall\hat{f}\in\mathbb{H}_f, \end{cases} \tag{33}$$

where for every $\hat{f}$, $\underline{v}_{\hat{f}} = \underline{v}_f(\hat{f})$, $p_{\hat{f}} = p_f(\hat{f})$ denote the solution of the system given by the first equation in (33) corresponding to a right end side

$f = \hat{f}$. The system (30) becomes: find $\underline{\Phi_f} = (\underline{v}_f, p_f) \in Y \times \mathbb{H}^p, \underline{Q} = (\underline{q}, \sigma) \in W \times \mathbb{H}^p$ s.t.

$$\begin{cases} \mathcal{L}(\underline{\Phi_f}, \underline{\hat{\Phi}}) = B(f, \underline{\hat{\Phi}})\ \forall \underline{\hat{\Phi}} \in W \times \mathbb{H}^p, \\ \mathcal{L}^*(\underline{Q}, \underline{\hat{W}}) = \langle J'_\Phi(f, \underline{\Phi}), \underline{\hat{W}} \rangle\ \forall \underline{\hat{W}} \in Y \times \mathbb{H}^p, \\ \alpha(f, \hat{f})_{\mathbb{H}_f} + B(\hat{f}, \underline{Q}) + \\ +\gamma_1 \int_0^T \int_{\Omega_0} m_{wd}(\nabla \times \underline{v}_f + m_1 \mathcal{R}_f f - \mathcal{R}_{obs,1}) m_1 \mathcal{R}_f \hat{f} d\Omega dt = 0\ \forall \hat{f} \in \mathbb{H}_f, \end{cases} \tag{34}$$

where

$$\langle J'_\Phi(f, \underline{\Phi}), \underline{\hat{W}} \rangle = \gamma_1 \int_0^T \int_{\Omega_0} m_{wd}(\nabla \times \underline{v}_f + m_1 \mathcal{R}_f f - \mathcal{R}_{obs,1}) \cdot (\nabla \times \underline{\hat{q}}) d\Omega dt +$$

$$+\gamma_2 \int_0^T \int_{\Gamma_{out}} (\underline{v}_f - \underline{v}_{out}) \cdot \underline{\hat{q}} d\Gamma dt + \gamma_2 \int_0^T \int_{\Gamma_{out}} (p_f - p_{out}) \hat{\sigma} d\Gamma dt +$$

$$+\gamma_3 \int_0^T \int_{\Gamma_{out}} \mathcal{J}(p_f - p_{out}) \cdot \mathcal{J} \hat{\sigma} d\Gamma dt.$$

Consider now the problem (34) for $\alpha > 0$.

Proposition 6.1 *For any $\alpha > 0$ problem (34) has a unique solution for each given $\underline{\mathcal{R}}_{obs,1}$.*

Proof. Following [1], we formally invert L and L^* in the first and second equations of (31) then we substitute $\underline{\Phi}$, $\underline{Q}$ into the third equation and we obtain the following weak problem: $f \in \overline{\mathbb{H}}_f$ satisfies:

$$\alpha(f, \hat{f})_{\mathbb{H}_f} + (Af, A\hat{f})_{\mathbb{L}^2(x_1,x_2)} = (G, A\hat{f})_{L^2(x_1,x_2)}\ \forall \hat{f} \in \mathbb{H}_f. \tag{35}$$

A is a linear operator, while G depends on the data. Precisely, from (33) we obtain:

$$(f, \hat{f})_{\mathbb{H}_f} = (\Lambda_f f, \hat{f})_{\mathbb{L}^2(x_1,x_2)},$$

$$(Af, A\hat{f})_{\mathbb{L}^2(x_1,x_2)} = \gamma_1 \int_0^T \int_\Omega m_{wd}(\nabla \times \underline{v} + m_1 \mathcal{R}_f f) \cdot (\nabla \times \underline{v}_{\hat{f}} + m_1 \mathcal{R}_f \hat{f}) d\Omega dt +$$

$$+\gamma_2 \int_0^T \int_{\Gamma_{out}} (p p_{\hat{f}} + \underline{v} \cdot \underline{v}_{\hat{f}}) d\Gamma dt + \gamma_3 \int_0^T \int_{\Gamma_{out}} \mathcal{J} p \cdot \mathcal{J} p_{\hat{f}} d\Gamma dt,$$

$$(G, A\hat{f})_{\mathbb{L}^2(x_1,x_2)} = \gamma_1 \int_0^T \int_\Omega m_{wd} \mathcal{R}_{obs,1} \cdot (\nabla \times \underline{v}_{\hat{f}} + m_1 \mathcal{R}_f \hat{f}) d\Omega dt +$$

$$+\gamma_2 \int_0^T \int_{\Gamma_{out}} (p_{out} p_{\hat{f}} + \underline{v}_{out} \cdot \underline{v}_{\hat{f}}) d\Gamma dt + \gamma_3 \int_0^T \int_{\Gamma_{out}} \mathcal{J} p_{out} \cdot \mathcal{J} p_{\hat{f}} d\Gamma dt,$$

where $\underline{\Phi} = (\underline{v}, p) = L^{-1}Bf$, $\underline{\Phi}_{\hat{f}} = (\underline{v}_{\hat{f}}, p_{\hat{f}}) = L^{-1}B\hat{f}, \forall \hat{f} \in \mathbb{H}_f$.
We see that if $\alpha > 0$ then problem (35) has unique solution which satisfies $\|f\|^2_{\mathbb{H}_f} \leq \|G\|^2/(2\alpha) < \infty$. Correspondingly we can construct $\underline{v}$, p, $\underline{q}$, σ, which together with f provides the unique solution of (34).

Consider now problem (34) with $\alpha = 0$.

Proposition 6.2 *Assume that:*
i) The solution of the generalized non stationary Stokes problem satisfies:

$$\left(\frac{\partial v_0}{\partial y}\right)^2 + \left(\frac{\partial u_0}{\partial y}\right)^2 > 0 \ at\ y = 0,\ x \in (x_1, x_2);$$

ii) *problem (34) has a solution in the class* $(L^2(0,T;H^2(\Omega)^2) \times L^2(0,T;H^1(\Omega)) \times W^{1,\infty}(x_1, x_2)$.
Then this solution is unique.
Proof. Let $(\underline{v}_1, p_1, f_1)$ and $(\underline{v}_2, p_2, f_2)$ be two solutions of (34). Then for $\underline{v} = \underline{v}_1 - \underline{v}_2, p = p_1 - p_2, f = f_1 - f_2$ from (33) we obtain:

$$\begin{cases} a_0(\underline{v}, \hat{\underline{v}}) - b_0(p, \hat{\underline{v}}) = F(f, \hat{\underline{v}}) \ \forall \hat{\underline{v}} \in W, \\ b_0(\hat{\underline{p}}, \underline{v}) + b_f(f; \hat{p}, \underline{v}_0) = 0 \ \forall \hat{p} \in \mathbb{H}^p, \\ \nabla \times \underline{v} + m_1 \mathcal{R}_f f = 0 \ in\ \Omega \times (0,T), \\ p = 0,\ \underline{v} = 0 \ on\ \Gamma_{out} \times (0,T). \end{cases} \tag{36}$$

Consider the classical form of the second and the third equation from (36) in $\Omega_{2,0} \times (0,T)$

$$\nabla \cdot \underline{v} = 0,\ \nabla \times \underline{v} = 0 \ in\ \Omega_{2,0} \times (0,T).$$

Then $\Delta \underline{v} = 0$ in $\Omega_{2,0}$ $\forall t \in (0,T)$. Considering $\hat{\underline{v}}$ with $supp(\hat{\underline{v}}) \subseteq \Omega_{2,0}$ from the first equation of (36) we find $\nabla p = 0$, then $p = const$ in $\Omega_{2,0}$ and $-p \cdot \underline{n} + \nu \frac{\partial \underline{v}}{\partial \underline{n}} = 0$ *on* Γ_{out} $\forall t$. Since $p = 0$ *on* Γ_{out} then $p = 0$ *in* $\Omega_{2,0}$ and $\nu \frac{\partial \underline{v}}{\partial \underline{n}} = 0$ *on* Γ_{out} too. Consequently, for all $t \in (0,T)$, $\underline{v}$ satisfies:

$$\Delta \underline{v} = 0 \ in\ \Omega_{2,0},\ \underline{v} = \nu \frac{\partial \underline{v}}{\partial \underline{n}} = 0 \ on\ \Gamma_{out}$$

Owing to the uniqueness continuation theorem this Cauchy problem has only the trivial solution $\underline{v} = 0$ *in* $\Omega_{2,0}$. Since $\underline{v} \in L^2(0,T;H^2(\Omega)^2)$ then

$$\underline{v} = \frac{\partial \underline{v}}{\partial \underline{n}} = 0 \ on\ \Gamma_0 := \{(x,y) : y = 0, x_1 < x < x_2\},\ \forall t \in (0,T).$$

Consider now the second and third equations from (36) in $\Omega_{1,0}$ in their classical form, $\forall\ t\ \in\ (0,T)$:

$$\begin{cases} \nabla \cdot \underline{v} - \left[y\left(\frac{f_x f_0 - f_{0,x} f}{f_0^2}\right)\frac{\partial u_0}{\partial y} + \frac{f}{f_0}\frac{\partial v_0}{\partial y}\right] = 0\ in\ \Omega_{1,0}, \\ \nabla \times \underline{v} - \left[y\left(\frac{f_x f_0 - f_{0,x} f}{f_0^2}\right)\frac{\partial v_0}{\partial y} - \frac{f}{f_0}\frac{\partial u_0}{\partial y}\right] = 0\ in\ \Omega_{1,0}. \end{cases} \tag{37}$$

On Γ_0 we have:

$$\nabla \cdot \underline{v} - \frac{f}{f_0}\frac{\partial v_0}{\partial y} = 0,\ \nabla \times \underline{v} + \frac{f}{f_0}\frac{\partial u_0}{\partial y} = 0,$$

$$|f(x)| = f_0 \frac{\left[(\nabla \cdot \underline{v})^2 + (\nabla \times \underline{v})^2\right]^{1/2}}{\left[\left(\frac{\partial v_0}{\partial y}\right)^2 + \left(\frac{\partial u_0}{\partial y}\right)^2\right]^{1/2}}\ on\ \Gamma_0,$$

(the dependence of the right end side on x and y is understood). Since $\underline{v} = \frac{\partial \underline{v}}{\partial \underline{n}} = \frac{\partial \underline{v}}{\partial y} = 0\ on\ \Gamma_0$, then

$$\nabla \cdot \underline{v}|_{y=0} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}|_{y=0} = 0,\ \nabla \times \underline{v}|_{y=0} = \frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}|_{y=0} = 0,\ x \in (x_1, x_2).$$

i.e. $f(x) = 0$. Therefore, $\underline{v} = 0,\ p = 0$ too, $\forall t \in (0,T)$.

7 ITERATIVE PROCESSES

In this section we propose some iterative processes which are well suited for solving the space-time variational equations obtained in the previous sections. Consider the problem (31); if for $k = 0, 1, \dots$ $f^{(k)}$ is known then $f^{(k+1)}$ can be determinated by solving the following equations [1]:

$$\begin{cases} L\Phi^{(k)} = Bf^{(k)}, \\ L^* \underline{Q}^{(k)} = \Lambda_w J_\Phi(f^{(k)}, \underline{\Phi}^{(k)}), \\ \Lambda_c w^{(k)} = B^* \underline{Q}^{(k)} + \Lambda_f J_f(f^{(k)}, \underline{\Phi}^{(k)}), \\ f^{(k+1)} = f^{(k)} - \tau_k(\alpha f^{(k)} + w^{(k)}), \end{cases} \tag{38}$$

where $\{\tau_k\}$ is a family of parameters whose determination follows from the theory of extremal problems [18], the general theory of iterative processes (see [6], [11], [12]), and the ill-posed problems theory ([13] and [16]). Its variational form reads as:

$$\begin{cases} a_0(\underline{v}^{(k)}, \hat{\underline{v}}) - b_0(p^{(k)}, \hat{\underline{v}}) = F(f^{(k)}, \hat{\underline{v}}) \ \forall \hat{\underline{v}} \in W, \\ b_0(\hat{\underline{p}}, \underline{v}^{(k)}) = -b_f(f^{(k)}; \hat{p}, \underline{v}_0) \ \forall \hat{p} \in \mathbb{H}^p, \\ a_0(\hat{\underline{q}}, \underline{q}^{(k)}) + b_0(\sigma^{(k)}, \hat{\underline{q}}) = G_k(\hat{\underline{q}}) \ \forall \hat{\underline{q}} \in Y, \\ -b_0(\hat{\sigma}, \underline{q}^{(k)}) = g_k(\hat{\sigma}) \ \forall \hat{\sigma} \in \mathbb{H}^p, \\ (w^{(k)}, \hat{f})_{\mathbb{H}_f} = d_k(\hat{f}) \ \forall \hat{f} \in \mathbb{H}_f, \\ f^{(k+1)} = f^{(k)} - \tau_k(\alpha f^{(k)} + w^{(k)}), \quad k = 0, 1, \dots, \end{cases} \tag{39}$$

where

$$F(f^{(k)}, \hat{\underline{v}}) = b_f(f^{(k)}, p_0, \hat{\underline{v}}) + G_1(f^{(k)}, \hat{\underline{v}}) - a_f(f^{(k)}, \underline{v}_0, \hat{\underline{v}}),$$

$$G_k(\hat{\underline{q}}) = \gamma_1 \int_0^T \int_{\Omega_0} m_{wd}(\nabla \times \underline{v}^{(k)} + m_1 \mathcal{R}_f f^{(k)} - \mathcal{R}_{obs,1}) \cdot (\nabla \times \hat{\underline{q}}) d\Omega dt +$$

$$+\gamma_2 \int_0^T \int_{\Gamma_{out}} (\underline{v}^{(k)} - \underline{v}_{out}) \cdot \hat{\underline{q}} d\Gamma dt,$$

$$g_k(\hat{\sigma}) = \gamma_2 \int_0^T \int_{\Gamma_{out}} (p^{(k)} - p_{out}) \hat{\sigma} d\Gamma dt + \gamma_3 \int_0^T \int_{\Gamma_{out}} \mathcal{J}(p^{(k)} - p_{out}) \cdot \mathcal{J} \hat{\sigma} d\Gamma dt,$$

$$d_k(\hat{f}) = F(\hat{f}, \underline{q}^{(k)}) - b_f(\hat{f}; \sigma^{(k)}, \underline{v}_0) +$$

$$+\gamma_1 \int_0^T \int_{\Omega_0} m_{wd}(\nabla \times \underline{v}^{(k)} + m_1 \mathcal{R}_f f^{(k)} - \mathcal{R}_{obs,1}) m_1 \mathcal{R}_f \hat{f} d\Omega dt.$$

Consider now the *finite dimensional case* in which the function $f, \{f^{(k)}\}, \hat{f}$ all are sought for in a finite-dimensional subspace $\mathbb{H}_{f,N} \subset \mathbb{H}_f$ of dimension $N < \infty$, whose basis $\varphi_i \in W^{1,\infty}(x_1, x_2), i = 1, 2, \dots, N$. Then the following theorem holds true.

Theorem 7.1 *Assume that:*

$$\Omega_{wd} = \Omega,$$

$$\left(\frac{\partial v_0}{\partial y}\right)^2 + \left(\frac{\partial u_0}{\partial y}\right)^2 > 0 \ at \ y = 0, \ x \in (x_1, x_2).$$

Then:

1. *The problem (33) is well posed (i.e. it admits a unique solution that depends continuously on the data) for $\alpha \geq 0$ and any $N < \infty$;*

2. *The iterative process (39) is convergent for any $\alpha > 0$, $N < \infty$, provided the parameters $\tau_k > 0$, $k = 0, 1, 2, \dots.$ are small enough;*

3. *If α is sufficiently small while k is sufficiently large, then $\{\underline{v}^{(k)}, p^{(k)}, f^{(k)}\}$ can be regarded as an approximate solution of problem (33).*

Proof:

1. The existence of the solution for $\alpha > 0$ has been proved earlier in Proposition 6.1. Let us consider the case $\alpha = 0$. Since $f = \Sigma_{i=1}^{N} a_i \varphi_i \in \mathbb{H}_{f,N}$ then in the form (35) with $\alpha = 0$ we conclude that this problem is well posed (because problem (33) can have only unique solution in $\mathbb{X} \times \mathbb{H}^p \times \mathbb{H}_f$, see Proposition 6.2). We assume the unsteady Stokes problem to be well posed for given $f \in \mathbb{H}_f$. Hence the problem (33) is well posed too.

2. If $\alpha > 0$ then the bilinear form on the left hand side of (35) is coercive and continuous with respect to the norm

$$\|f\|_{A,\alpha} = \sqrt{\alpha \|f\|^2_{\mathbb{H}_f} + \|Af\|^2_{\mathbb{L}^2(x_1,x_2)}}.$$

Then the process given by

$$(f^{(k+1)}, \hat{f})_{\mathbb{H}_f} = (f^{(k)}, \hat{f})_{\mathbb{H}_f} - \tau(\alpha(f^{(k)}, \hat{f})_{\mathbb{H}_f} + (Af^{(k)}, A\hat{f})_{\mathbb{L}^2(x_1,x_2)}) -$$

$$-(G, A\hat{f})_{L^2(x_1,x_2)}, \; k = 0, 1, \ldots$$

is convergent for small $\tau > 0$. Hence also the process (39) is convergent and

$$\|\underline{v}^{(k)} - \underline{v}\|_Y + \|p^{(k)} - p\|_{\mathbb{H}^p} + \|f - f^{(k)}\|_{\mathbb{H}_f} \to 0, \; k \to \infty. \tag{40}$$

If $\Lambda_C^{-1} A^* A \in [C_1, C_2], C_1, C_2 = const$, choosing $\tau_k = 2/(2\alpha + C_1 + C_2)$ we obtain (see [1]):

$$\|\underline{v}^{(k)} - \underline{v}\|_Y + \|p^{(k)} - p\|_{\mathbb{H}^p} + \|f - f^{(k)}\|_{\mathbb{H}_f} \leq C\Big(\frac{C_2 - C_1}{2\alpha + C_1 + C_2}\Big)^k \tag{41}$$

which tends to zero as $k \to \infty$.

3. Let $\underline{v}_0, \; p_0, \; f_0$ be a solution of (33) when $\alpha = 0$. According to the theory of ill-posed problems ([13] and [16]) we have: $\|f_0 - f_\alpha\|_{\mathbb{H}_f} \to 0$ as $\alpha \to +0$, where $(f_\alpha, \underline{v}_\alpha, p_\alpha)$ is the solution of (33) for $\alpha > 0$. Hence

$$\|\underline{v}_0 - \underline{v}_\alpha\|_Y + \|p_0 - p_\alpha\|_{\mathbb{H}^p} \to 0, \; as \; \alpha \to +0.$$

Owing to (40) this concludes our proof.

8 TEST PROBLEM AND NUMERICAL RESULTS

To test our methodology we consider some test problems on simplified configurations. Wall curvature was considered only in the zone of the incoming branch of the bypass ($-2 \leq x \leq 0$) where we set $f_0 = 1 - sin(x\pi/4)$; in the remaining parts we used piecewise constant function. The graft angle of the bypass incoming branch (which influences vorticity) has been set equal to zero (between the artery and the new incoming branch there isn't a relative angle).
Velocity values $\underline{v}_{in}$ at the inflow are chosen in such a way that the Reynolds number $Re = \frac{\bar{v} \cdot D}{\nu}$ has order 10^3, the mean Reynolds number is 1250, the maximum is 2500. The inlet Poiseuille velocity profile has a pulsatile nature over the period $T = 1s$ (heart beat) and the law considered was: $\underline{v}_{in} = -0.475(y-1)(y-2)(1-t)\underline{n}$, see Fig.(5). Blood kinematic viscosity $\nu = \frac{\mu}{\rho}$ is equal to $4 \cdot 10^{-6}\ m^2\ s^{-1}$, blood density $\rho = 1\ g\ cm^{-3}$ and dynamic viscosity $\mu = 4 \cdot 10^{-2}\ g\ cm^{-1} s^{-1}$; $\bar{v}$ is the mean inflow velocity $\overline{v} = \left(\frac{\int_{\Gamma_{in}} \int_0^T |\underline{v}_{in} \cdot \underline{n}| d\Gamma dt}{\int_{\Gamma_{in}} \int_0^T d\Gamma dt} \right)$ which yields the desired Reynolds number, while D is the arterial diameter (3.5 mm), see [9] and [10].
In this section we present numerical results using as cost functional $J(f, \underline{v}) = J_1(f, \underline{v})$ (introduced in Eq.(24)). This is equivalent to the $\mathbb{L}^2$ norm of the vorticity on $(\Omega_{wd} \times (0,T))$ (restricted in the downfield zone of the new incoming branch of the bypass, where the observation is made). We have set $\mathcal{R}_{obs,1} = 0.45 \mathcal{R}(f_0)\underline{v}$.
For the space approximation of the Stokes equation we use $P^1 - P^1$ (piecewise linear) finite elements and SUPG stabilization (see [11]). Time discretization is based on first order backward differentiation (which is unconditionally stable).
Figs.(6)-(11) give an account of the numerical results obtained and show how the shape of the bypass is changed to reduce downfield vorticity. The shape is smoothed out at the upper corner and a slightly cuffed incoming branch is created. These results provides a shape which resembles the Taylor patch (see [3]). We have shown original bypass configuration (and horizontal velocity to show relevant fluid dynamics phenomena) at different time step ($t_1 = 0.1$ s, $t_2 = 0.4$ s, $t_3 = 0.7$ s) and then the configurations obtained at different steps of our optimization process (i.e. at different iterations $N_1 = 5$, $N_2 = 11$ and $N_3 = 17$). These results can be regarded as an improvement of previous results that were obtained in [2] using a steady fluid flow model. The similarity between the present results and those obtained in [2] can be ascribed to the fact that shape $f^{(k)}$ doesn't depend on time, moreover the shape variation at each optimization step

$\delta f^{(k)}$ is given by the sum of the contributions from all time-steps in $(0, T)$ and the first contribution is the one that dominates (when considering a pulsatile flow). Fig.(12) shows distributed (pointwise) vorticity ($[1/s]$) in bypass configuration before and at the end of the control process. Vorticity is diminished near the upper corner (in the original configuration we have a concentrated high value of it at the singularity) but also in the bed of the artery in the downfield zone: this indicates that the flow is less disturbed and the main flow decreases its attitude to recirculate in the stenosed zone. These phenomena are due to the fact that the bypass section is smoothed and increased and the flow is guided more smoothly through the section. Fig.(13) shows the variation of corrections $(x, \delta f^n(x)y/\beta)$ at the first iteration and after 14 iterations of the shape design process. These plots represent a measure of shape variation (corrections) with respect to the problem (δf^n is related with state and adjoint solutions). At the beginning of the process we can see (plot on the left) that the corner is the most sensible zone of the domain, after 14 iterations of the process (plot on the right) the shape variation is reduced (max $\delta f^{(n)}$ is 10% of the one at the first step). In Fig.(14) we report total vorticity ($[m^2/s]$) in downfield zone during the control process at different time steps. We underline that only for the curve at $t = 1$ s we have a complete result dealing with vorticity reduction over a period T, other curves represent partial results considering a fraction of T (optimization is carried out over a period T at each iteration). The most important contribution to total vorticity is given by the flow behaviour in the first part of the period considered. Vorticity (total) reduction is quite substantial. Fig.(14) shows the total vorticity trend in time. At the end of the process we can see that the vorticity behaviour in time has the same trend, however its value is reduced.

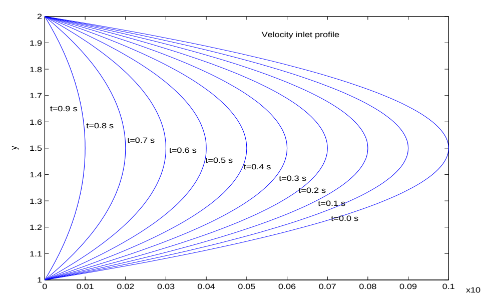

Figure 5: Unsteady (pulsatile) Stokes velocity profiles at the inflow $[ms^{-1}]$.

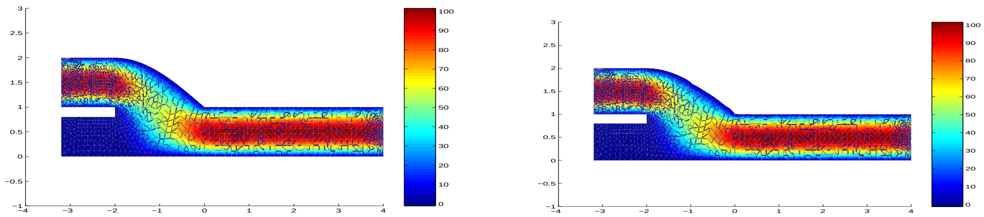

Figure 6: Horizontal velocity $[cms^{-1}]$ at $t = 0.1s$ for initial test configuration (left) and after 5 iterations (right).

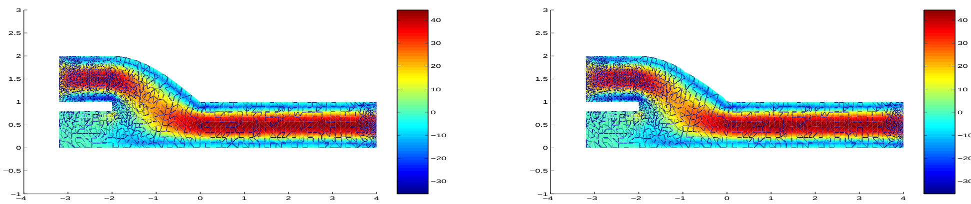

Figure 7: Horizontal velocity $[cms^{-1}]$ at $t = 0.4s$ for initial test configuration (left) and after 5 iterations (right).

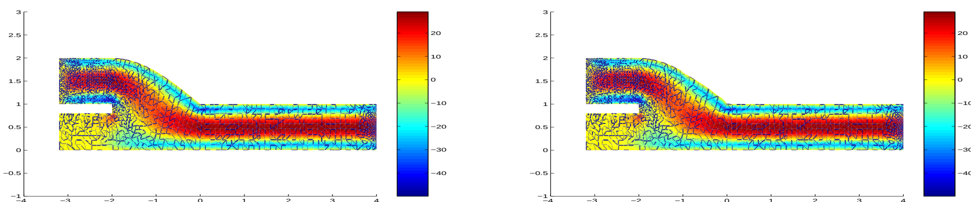

Figure 8: Horizontal velocity $[cms^{-1}]$ at $t = 0.7s$ for initial test configuration (left) and after 5 iterations (right).

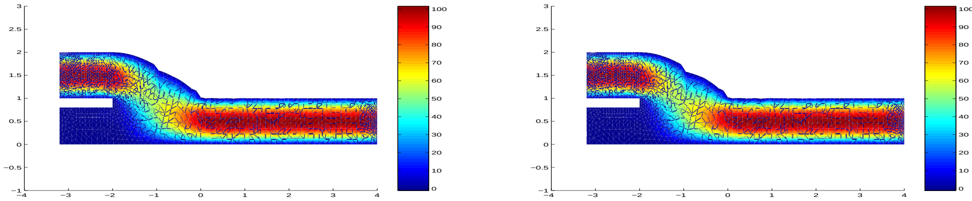

Figure 9: Horizontal velocity $[cms^{-1}]$ after $t = 0.1s$ at 11 (left) and 17 (right) iterations.

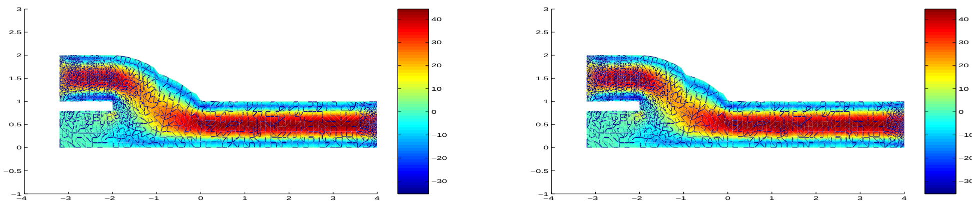

Figure 10: Horizontal velocity $[cms^{-1}]$ after $t = 0.4s$ at 11 (left) and 17 (right) iterations.

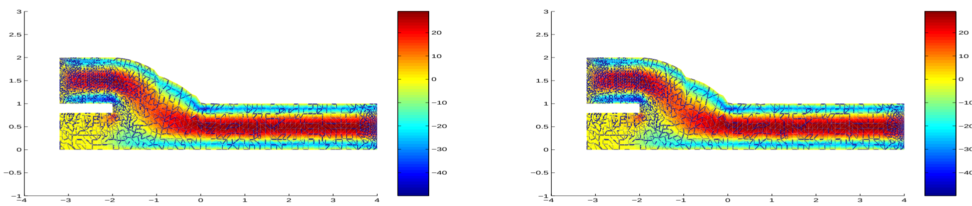

Figure 11: Horizontal velocity $[cms^{-1}]$ after $t = 0.7s$ at 11 (left) and 17 (right) iterations.

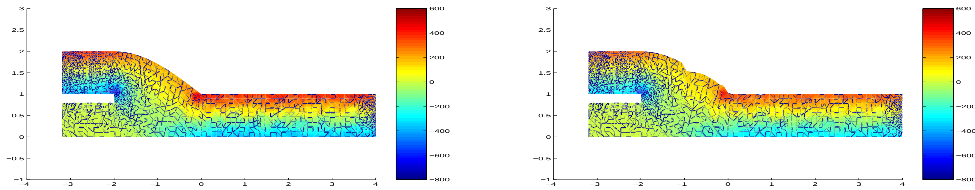

Figure 12: Distributed vorticity $[s^{-1}]$ in original configuration (left) and at the end of the optimization process (right). Vorticity in the upper corner and in the bed of the artery is diminished.

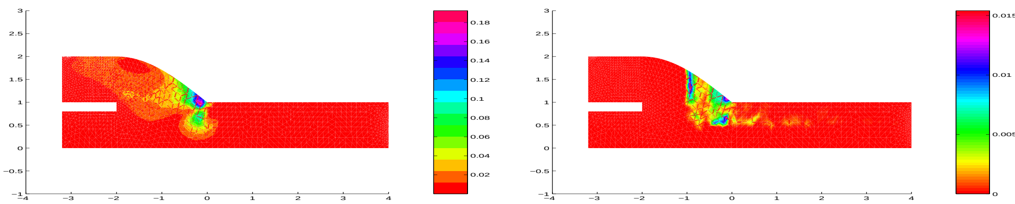

Figure 13: Variation of corrections $(x, \delta f^n(x)y/\beta)$ at the first iteration (left) and after 14 iterations of the process (right).

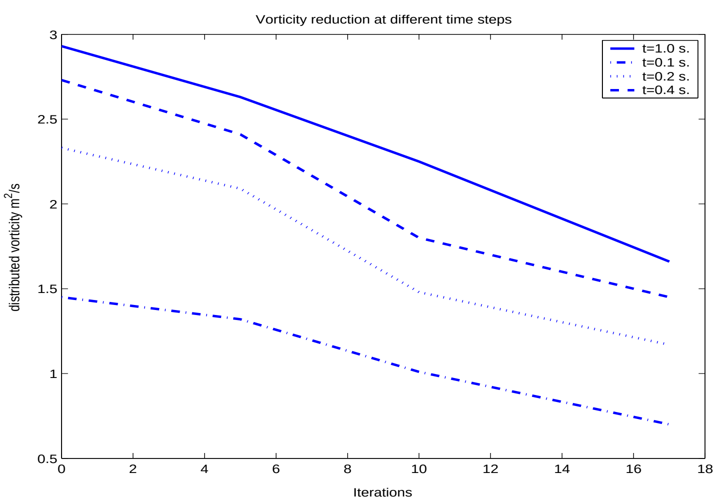

Figure 14: Total vorticity $[m^2 s^{-1}]$ reduction at different time steps during shape optimization.

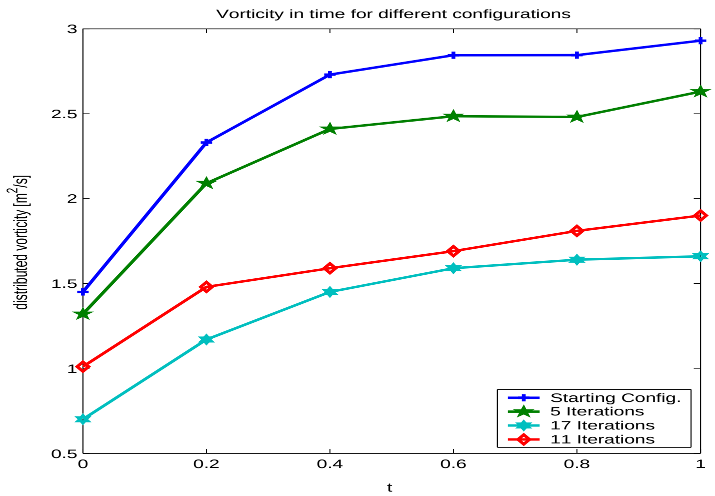

Figure 15: Total vorticity $[m^2 s^{-1}]$ in time during shape optimization for different configurations.

9 CONCLUSIONS

In this study we have faced the problem of determining the first corrections for the shape design of simplified two-dimensional bypass configurations.
When solving this problem we aim at determining a new boundary shape in such a way that the corresponding flow field in the new bypass region fulfills a suitable optimality criterium expressed by functions that are associated to indexes of clinical relevance. Precisely we have introduced a method for computing $f = f_0 + \varepsilon f_1$, where f_0 describes the initial configuration of the bypass boundary and f_1 the so-called first correction. Once the latter is known, we can restart the procedure with this new value as f_0, and look for another first correction, and so on.
In the future, optimal control and shape optimization applied to possibly three-dimensional fully unsteady incompressible Navier-Stokes equations accounting for fluid-structure interaction problem could provide more realistic design indications concerning surgical prosthesis realizations.
A further development will concern the set up of efficient schemes for reduced-basis methodology approximations (see for example [7] and [15]) which could be more efficient for use in a repetitive design environment as optimal shape design methodology requires. In [14] we present the state of the art of the problem into a more complex (multilevel optimization) framework.

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