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Mathematical Models in Science and Engineering

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Mathematical modeling aims to describe the different aspects of the real world, their interaction, and their dynamics through mathematics. It constitutes the third pillar of science and engineering, achieving the fulfillment of the two more traditional disciplines, which are theoretical analysis and experimentation. Nowadays, mathematical modeling has a key role also in fields such as the environment and industry, while its potential contribution in many other areas is becoming more and more evident. One of the reasons for this growing success is definitely due to the impetuous progress of scientific computation; this discipline allows the translation of a mathematical model—which can be explicitly solved only occasionally—into algorithms that can be treated and solved by ever more powerful computers. See Figure 1 for a synthetic view of the whole process leading from a problem to its solution by scientific computation. Since 1960 numerical analysis—the discipline that allows mathematical equations (algebraic, functional, differential, and integrals) to be solved through algorithms—had a leading role in solving problems linked to mathematical modeling derived from engineering and applied sciences. Following this success, new disciplines started to use mathematical modeling, namely information and communication technology, bioengineering, financial engineering, and so on. As a matter of fact, mathematical models offer new possibilities to manage the increasing complexity of technology, which is at the basis of modern industrial production. They can explore new solutions in a very short time period, thus allowing the speed of innovation cycles to be increased. This ensures a potential advantage to industries, which can save time and money in the development and validation phases. We can state therefore that mathematical modeling and scientific computation are gradually and relentlessly expanding in manifold fields, becoming a unique tool for qualitative and quantitative analysis. In the following paragraphs we will discuss the role of mathematical modeling and of scientific computation in applied sciences; their importance in simulating, analyzing, and decision making; and their contribution to technological progress. We will show some results and underline the perspectives in different fields such as industry, environment, life sciences, and sports.

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Scientific Computation for Technological Innovation

Linked to the incredible increase of computer calculation speed, scientific computation may be decisive enough to define the border between complex problems that can be treated and those that, on the contrary, cannot. The aim of scientific computation is the development of versatile and reliable models, detailed in closed form, and tested on a wide range of test cases, either analogical or experimental, for which there are helpful reference solutions.

A mathematical model must be able to address universal concepts, such as, for instance, the conservation of mass or the momentum of a fluid, or the moment of inertia of a structure; moreover, in order to obtain a successful numerical simulation, it is necessary to define which level of detail must be introduced in the different parts of a model

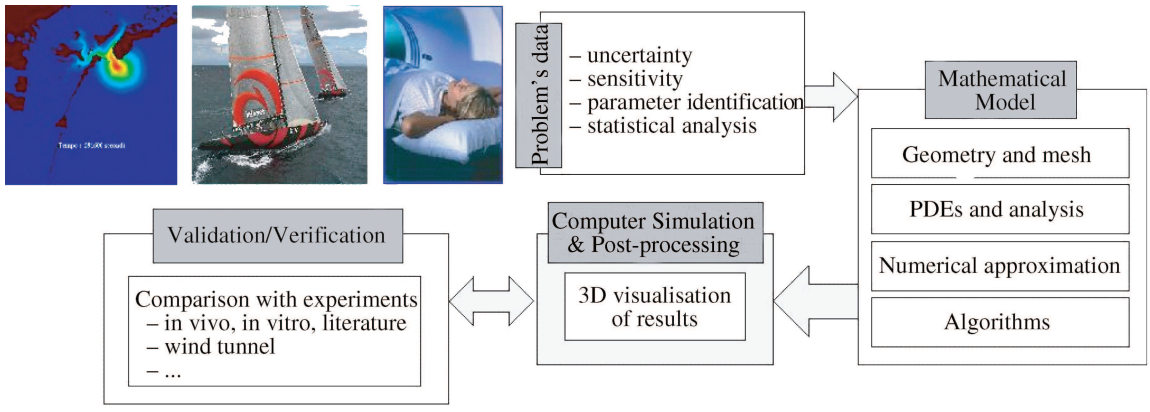

Figure 1. Scientific computing at a glance.

and which simplifications must be carried out to facilitate its integration into different models. Models able to simulate very complex problems should take into account uncertainty due to the lack of data (or data affected by noise) that feed the model itself. These kinds of models will be used to foresee natural, biological, and environmental processes, in order to better understand how complex phenomena work, and also to contribute to the design of innovative products and technologies.

An important aspect of scientific computation is represented by computational fluid dynamics (CFD), a discipline that aims to solve by computers problems governed by fluids. In aerospace, for example, CFD can be applied in many ways. Numerical models based on potential flow equations or on the more sophisticated Euler or Navier-Stokes equations can be used, for example, in the aerodynamic analysis of wing tips or for the whole fuselage for performance optimization. See Figure 2 and Figure 3 for numerical simulations carried out on, respectively, a civil aircraft (the Falcon 50) and the X29 experimental aircraft using the Euler equations solved by a stabilized finite element approximation [1]. Simulation implies validation and optimization, with the aim of designing aircraft able to meet certain requirements: better structural reliability, better aerodynamic performance, lower environmental impact thanks to the reduction in noise emissions (in the case of commercial airplanes), speed optimization, and improvement of maneuverability (in the case of military aircraft). The solution to these problems requires multi-objective optimization algorithms: deterministic, stochastic, or genetic. Moreover, models of electromagnetic diffusion are used to simulate external electromagnetic fields in order to restrain them from interfering with those generated by the several electronic circuits that are contained in the instrumentation on board. Models are used to simulate the stresses and the deformation of some parts of the aircraft (for the simulation of the analysis of materials strain), through algorithms for the interaction between fluid and structure with the aim of improving structural and dynamic stability. Similar analyses are studied in the car industry, where numerical simulation is used in virtually every aspect of design and car production. Models are used to simulate internal engine combustion in order to save fuel, improve the quality of emissions, and reduce noise. Moreover, to improve performance, security, and comfort, several kinds of equations must be solved, such as those modeling external and internal fluid dynamics, aero-elasticity, and aero-acoustic vibration dynamics, but also those governing thermal exchange, combustion processes, shock waves (occurring during the opening phase of an air bag), structural dynamics under large stresses, and large deformations to simulate the consequences of car crashes. The chemical industry uses mathematical models to simulate polymerization processes, pressing, or extrusion for complex rheologic materials, where the typical macro analysis of continuum mechanics must be connected to the micro one, the latter being more adequate to describe the complex rheology of materials with nanostructure. This requires the development of multiscale analysis techniques and algorithms, which are able to describe the exchange of mechanical, thermal, and chemical processes in heterogeneous spatial scales. In the electronics industry, the simulation of drift-diffusion, hydrodynamics, Boltzmann, or Schrödinger equations plays a key role in designing ever smaller and faster integrated circuits, with growing functionality and with dramatic waste reduction (which are fundamental, for example, in different applications of mobile phones). Efficient algorithms are

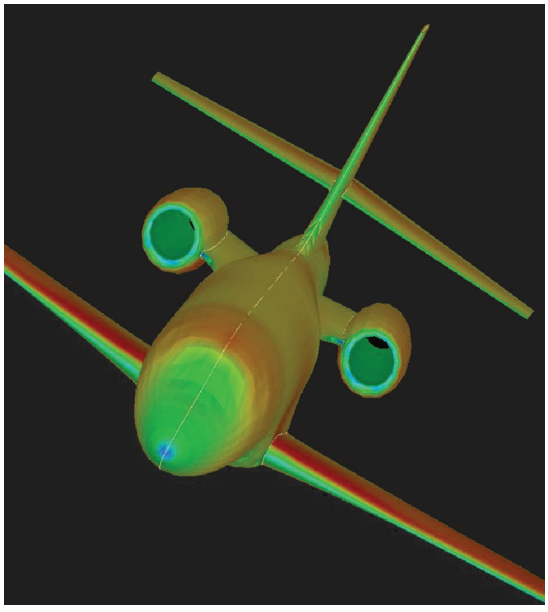

Figure 2. Mach number distribution and streamlines for a civil aircraft.

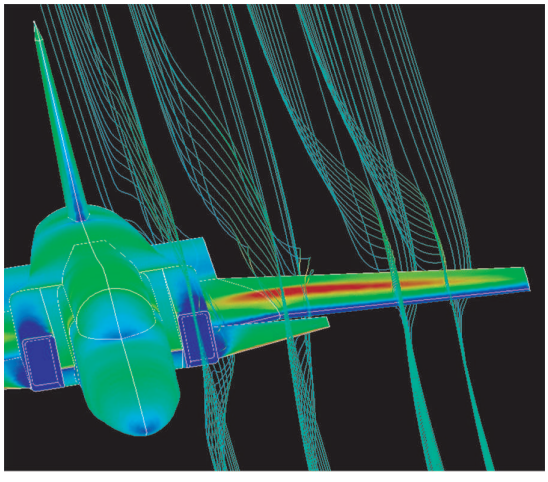

Figure 3. Mach number and streamlines on the X29 experimental aircraft.

useful also for coding and decoding multi-user messages.

Modeling the Weather

In the last few decades, the critical problem of predicting the weather in a short time (daily or weekly) has become more and more linked to long-term prediction (for a decade or even a century), to climatic evolution, and to atmospheric pollution problems. Luckily, there are natural climatic changes in a particular area that obey physical law, and can thus be simulated through mathematical models. Also, from a global point of view (over

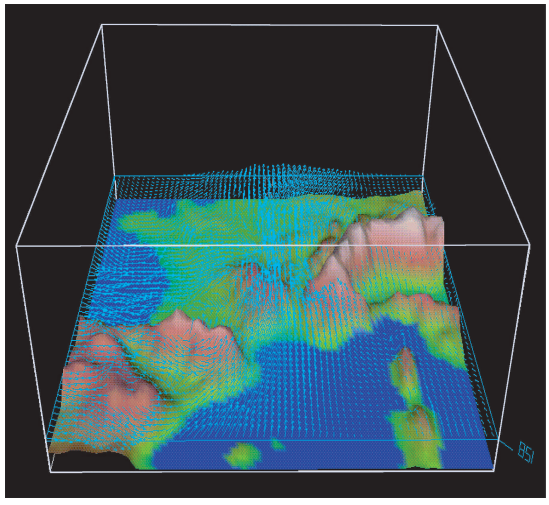

Figure 4. Wind velocity simulation over the Mediterranean Sea.

either a continental or worldwide scale) there are changes due to deterministic phenomena, for example to variation in the inclination of the earth's axis, the eccentricity of the earth's orbit, the oceanic circulation, or intense geological phenomena like volcanic eruptions. The meteorological prediction problem was formulated as a mathematical issue only at the beginning of the twentieth century by the Norwegian mathematician Vilhelm Bjerkned, who described atmospheric motion using the Euler equations for perfect gas dynamics (well known at that time), suitably modified in order to take into account the action of the force of gravity and the earth's rotation. Unluckily, data regarding the atmosphere were available only in a few points, and they referred to heterogeneous variables and to different periods of time.

Moreover, Euler equations described an extremely wide range of atmospheric motions, which can take place on spatial and temporal scales that are very different from each other (feet instead of miles, seconds instead of days). The lack of data regarding some of these scales may generate spurious motions (which do not exist in nature) and reduce the prediction quality. A realistic description of meteorological phenomena cannot but take into account the prediction of water steam distribution, its changes (from liquid to gas), and consequent rainfall. The first attempt to solve this problem from a numerical point of view was carried out by Lewis Richardson, who succeeded in calculating a concrete example of the solution of atmospheric motion on a region as wide as the whole of North Europe. The results obtained by Richardson through extremely complicated hand calculations led to completely wrong predictions, though: as a matter of fact, at that time there was no theory able to dominate

the traps of the equations to be solved. The contribution of Carl-Gustaf Rossby, one of Richardson's students, was decisive enough to optimize the efforts made by Richardson. After immigrating to the USA in the 1920s, he contributed to founding the meteorological service for civil and military aviation during the Second World War. Among the indirect contributions he gave, the weather prediction made by the Americans for D-Day (June 4, 1944) can be included. The simplified mathematical models introduced by Rossby allowed the first meteorological prediction to be made with an electronic computer, resulting from cooperation between John von Neumann and Jules Charney, which started in Princeton in the 1940s. In particular, it was possible to make a prediction for the whole of the North of America through a simplified model that described the atmosphere as a unique fluid layer. Even though it took 24 hours to make a prediction for the following 12 hours on the only electronic computer available (ENIAC), the efforts of von Neumann and Charney showed for the first time that a prediction based only on a mathematical model could achieve the same results as those by an expert on meteorology of that time. The modern approach to numerical weather prediction was born. As a matter of fact, beyond the spectacular improvements in computer performance, there have also been radical improvements in the accuracy of mathematical prediction tools, the development of a theory on the predictability of chaotic dynamical systems, and an improvement in data assimilation techniques. In the 1970s, the systematic use of surveys made by satellites was introduced, and it constitutes nowadays the most relevant part of the data used to start numerical models. Since then, the impact of scientific and technological progress has been very important. For instance, the IFS global model of the European Center for Medium Range Weather Forecasts (ECMWF) uses a computation grid with an average spatial resolution of about 22km horizontal and 90km vertical. This allows part of the stratosphere to be included. This model can make a 10-day prediction in about 1 hour on a modern parallel supercomputer, even though 6 further hours, necessary to insert the data, must be added. The IFS model allows reliable predictions to be made for about 7.5 days on a continental scale in Europe. See Figure 4 for an example of weather prediction.

Models for Life Sciences

In the 1970s, in vitro experiments, and those on animals, represented the main approach to cardiovascular studies. Recently, the progress of computational fluid dynamics and the great improvements of computer performance produced

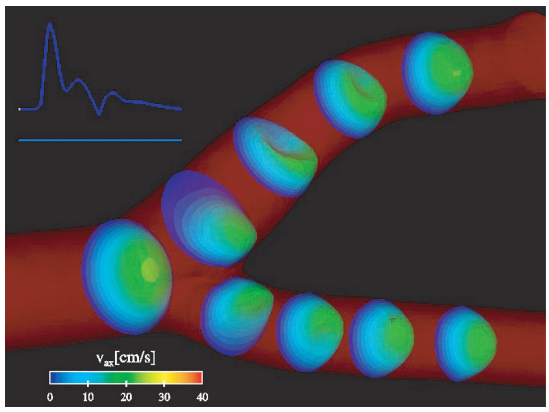

Figure 5. Computed velocity profiles downstream a carotid bifurcation.

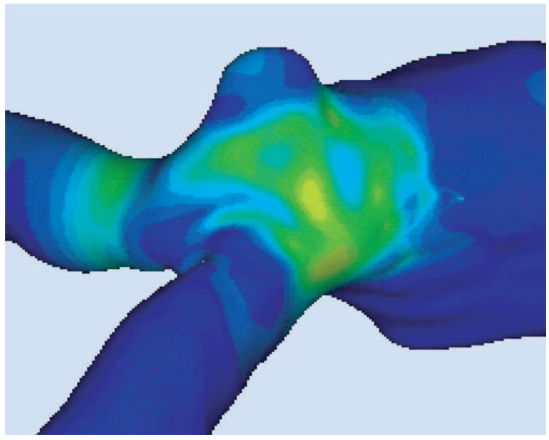

Figure 6. Shear stress distribution on a pulmonary artery.

remarkable advances that revolutionize vascular research [7].

For instance, a physical magnitude such as the shear stress on the endothelial membrane, which is very difficult to test in vitro, can be easily calculated on real geometries obtained with tri-dimensional algorithms, thanks to the support of modern and noninvasive data acquisition technology (such as nuclear magnetic resonance, digital angiography, axial tomography, and Doppler anemometry). Flowing in arteries and veins, blood mechanically interacts with vessel walls, generating complex fluid-structural interaction problems. As a matter of fact, the pressure wave transfers mechanical energy to the walls, which dilate; such an energy is returned to the blood flow while the vessels are compressed. Vascular simulation of the interaction between the fluid and the wall requires algorithms that describe both the energy transfer between the fluid (typically modeled by the Navier-Stokes equations) and the structure (modeled by

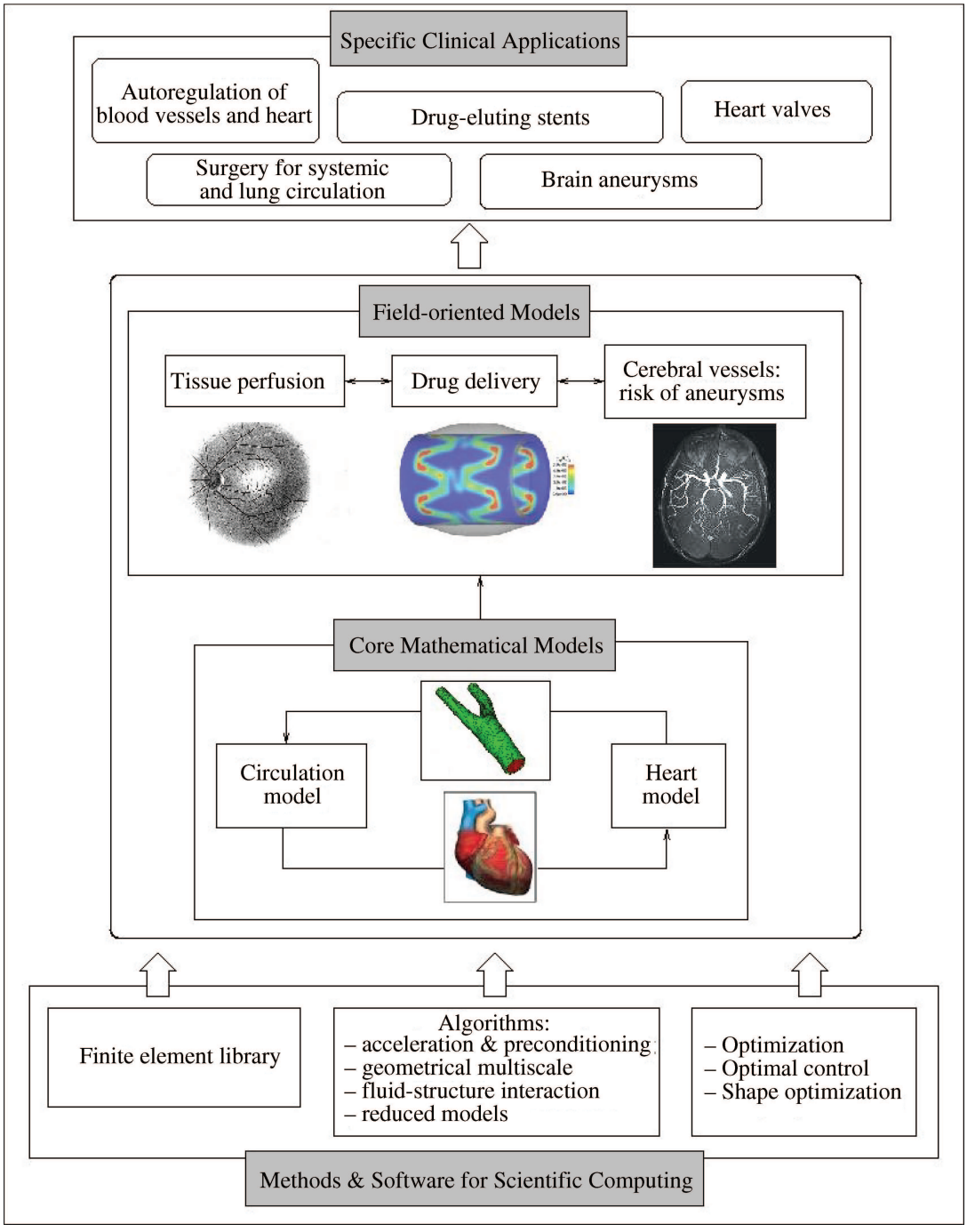

Figure 7. Scientific computing for cardiovascular flow simulation and related topics.

solid mechanics equations) at a macroscopic level, and the influence—at a microscopic level—of the shear stress on orientation, deformation, and damage of endothelial cells [8]. At the same time, flow equations must be coupled to appropriate models in order to describe the transport, diffusion, and absorption of chemical components in the blood (such as oxygen, lipids, and drugs), in the different layers that constitute artery walls (tunica intima, tunica media, and tunica adventitia). Numerical simulations of this kind may help to clarify biochemical modifications produced by changes in the flow field, generated, for example, by the presence of a stenosis, i.e., an artery narrowing. In the cardiovascular system, conditions of separated flow and secondary circulatory motions are met, not only in the presence of vessels featuring large curvature (e.g., the aortic bend or

the coronary arteries), but also downstream of bifurcations (for instance the carotid artery in its internal and external branches) or regions with restrictions due to the presence of stenosis. There are other areas with a flow inversion (from distal to proximal regions) and also areas with low shear stress with temporal oscillations [9]. These cases are nowadays recognized as potential factors in the development of arterial pathologies. A detailed comprehension of local haemodynamic change, of the effects of vascular wall modifications on the flow scheme, and of the gradual adaptation in the medium to long period of the whole system following surgical interventions, is nowadays possible thanks to the use of sophisticated computer simulations, and may be extremely useful in the preliminary phase before a therapeutic treatment. A similar scenario may provide specific data for surgical procedures. Simulating the flow in a coronary bypass, in particular the re-circulation that takes place downstream of the graft in the coronary artery, may help us to understand the effects of artery morphology on the flow and thus of the post-surgical progression. The theory of optimal shape control may be useful for designing a bypass able to minimize the vorticity produced downstream of the graft in the coronary artery. Similarly, the study of the effects of a vascular prosthesis and of implantation of artificial heart valves on local and global haemodynamics may progress thanks to more accurate simulations in the field of blood flow. In virtual surgery, the result of alternative treatments for a specific patient may be planned through simulations. This numerical approach is an aspect of a paradigm of practice, known as predictive medicine. See Figure 7 for a comprehensive picture on our current research projects in the field of cardiovascular flow simulations.

Models for Simulation and Competition

The application of mathematical models is not limited to the technological, environmental, or medical field. As a matter of fact, deterministic and stochastic models have been adopted for many years in analyzing the risk of financial products, thus facilitating the creation of a new discipline known as financial engineering. Moreover, the new frontier has already begun to touch sociology, architecture, free time, and sports.

As far as competitive sports are concerned, CFD for some years now has been assuming a key role in analyzing and designing Formula One cars. But Formula One racing is not the only field where mathematical/numerical modelling has been applied. As a matter of fact, my research group from EPFL has been involved in an extremely interesting experience, which saw the Swiss yacht Alinghi win the America's Cup both in 2003 and again in 2007.

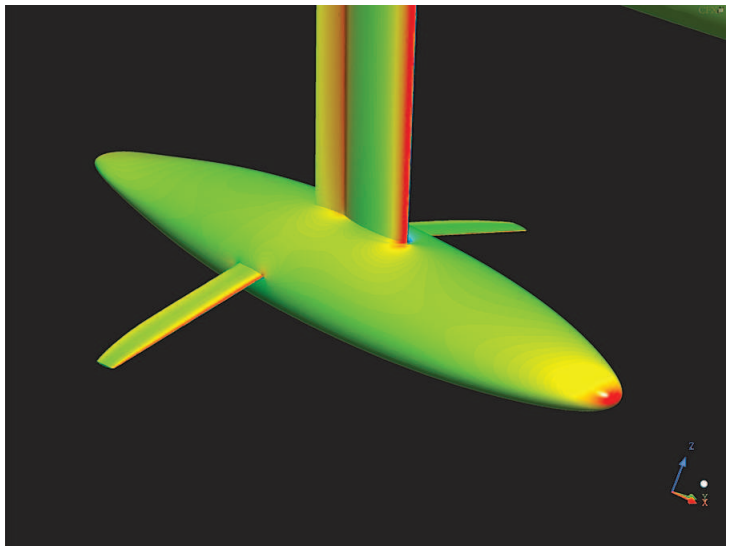

Figure 8. Pressure distribution around yacht appendages.

Until twenty years ago, the different designing teams used to develop different shapes of sails, hulls, bulbs, and keels. Nowadays the different geometric shapes have been standardized, and even the smallest details may make a difference from the results point of view. Quoting Jerome Milgram, a professor from MIT and an expert in advising different American America's Cup teams: "America's Cup teams require an extreme precision in the design of the hull, the keel, and the sails. A new boat able to reduce the viscous resistance by one percent, would have a potential advantage on the finish line of as much as 30 seconds." To optimize a boat's performance, it is necessary to solve the fluid-dynamics equations around the whole boat, taking into account the variability of wind and waves, of the different conditions during the yacht race, of the position, and of the moves of the opposing boat, but also the dynamics of the interaction between fluids (water and air) and the structural components (hull, appendages, sails, and mast) must be considered. Moreover, the shape and dynamics of the so-called free surface (the interface between air and water) has to be accurately simulated as well. A complete mathematical model must take into account all these aspects characterizing the physical problem. The aim is to develop together with the designers optimal models for the hull, the keel, and appendages.

Ideally, one wishes to minimize water resistance on the hull and appendages and to maximize the boost produced by the sails. Mathematics allows different situations to be simulated, thus reducing costs and saving time usually necessary to produce a great number of prototypes to be tested in

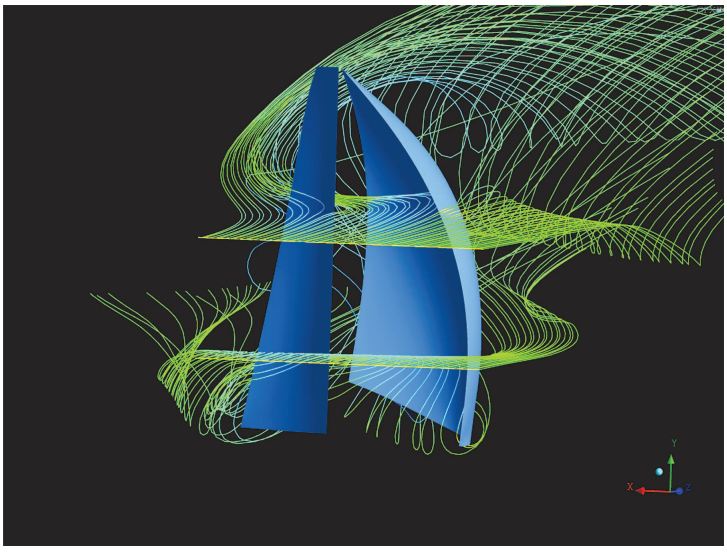

Figure 9. Streamlines around mainsails and spinnaker in a downwind leg.

a towing tank and wind tunnel. For each new boat simulation proposed by the designers (which were several hundred), it was necessary to build the geometrical model—about 300 splines surfaces are needed to overlay the whole boat—to create the grid on the surface of all the elements of the boat reliable enough to enable the determination of the transition between laminar flow and turbulent flow regions, and consequently to generate the volumetric grid in external domain. The Navier-Stokes equations for incompressible viscous flows must be used to describe both water and wind dynamics and the consequent free surface, which need to be completed by additional equations that allow the computation of turbulent energy and its dissipation rate. These equations cannot be solved exactly to yield explicit solutions in closed form. Their approximate solution requires the introduction of refined discretization methods, which allow an infinite dimensional problem to be transformed into a big but finite dimensional one. The typical calculation, based on finite volume schemes, involved the solution of nonlinear problems with many millions of unknowns. Using parallel algorithms, 24 hours on parallel calculation platforms with 64 processors were necessary to produce a simulation, characterized by more than 160 million unknowns. A further computation is concerned with the simulation of the dynamical interaction between wind and sails by fluid-structure algorithms. These simulations enable the design team to eliminate those solutions that seem innovative and to go on with those that actually guarantee better performance. Moreover, by simulating the effects of aerodynamic interaction between two boats, one can determine the consistency of shadow regions (the areas with less wind because of the position of a boat with respect to the other), the flow perturbation, and the turbulence vorticity generated by the interaction of the air, thus obtaining useful information for the tactician as well. These studies aim to design a boat having an optimal combination of the four features that an America's Cup yacht must have: lightness, speed, resistance, and maneuverability necessary to change the race outcome.

A more in-depth description of the mathematical tools necessary for this kind of investigation is provided in the next section.

Mathematical Models for America's Cup

The standard approach adopted in the America's Cup design teams to evaluate whether a design change (and all the other design modifications that this change implies) is globally advantageous, is based on the use of a Velocity Prediction Program (VPP), which can be used to estimate the boat speed and attitude for any prescribed wind condition and sailing angle. A numerical prediction of boat speed and attitude can be obtained by modeling the balance between the aerodynamic and hydrodynamic forces acting on the boat.

For example, on the water plane, a steady sailing condition is obtained imposing two force balances in the x direction (aligned with the boat velocity) and the y direction (normal to x on the water plane) and a heeling moment balance around the centerline of the boat:

$$\begin{aligned} D_h + T_a &= 0, \\ S_h + S_a &= 0, \\ M_h + M_a &= 0, \end{aligned} \tag{1}$$

where D_h is the hydrodynamic drag (along the course direction), T_a is the aerodynamic thrust, S_h is the hydrodynamic side force perpendicular to the course, S_a is the aerodynamic side force, M_h and M_a are, respectively, the hydromechanical righting moment and the aerodynamic heeling moment around the boat mean line. The angle β_Y between the course direction and the boat centerline is called *yaw* angle. The aerodynamic thrust and side force can be seen as a decomposition in the reference system aligned with the course direction of the aerodynamic lift and drag, which are defined on a reference system aligned with the apparent wind direction. Similar balance equations can be obtained for the other degrees of freedom.

In a VPP program, all the terms in system (1) are modeled as functions of boat speed, heel angle, and yaw angle. Suitable correlations between the degrees of freedom of the system and the different force components can be obtained based on different sources of data: experimental results, theoretical predictions, and numerical simulations.

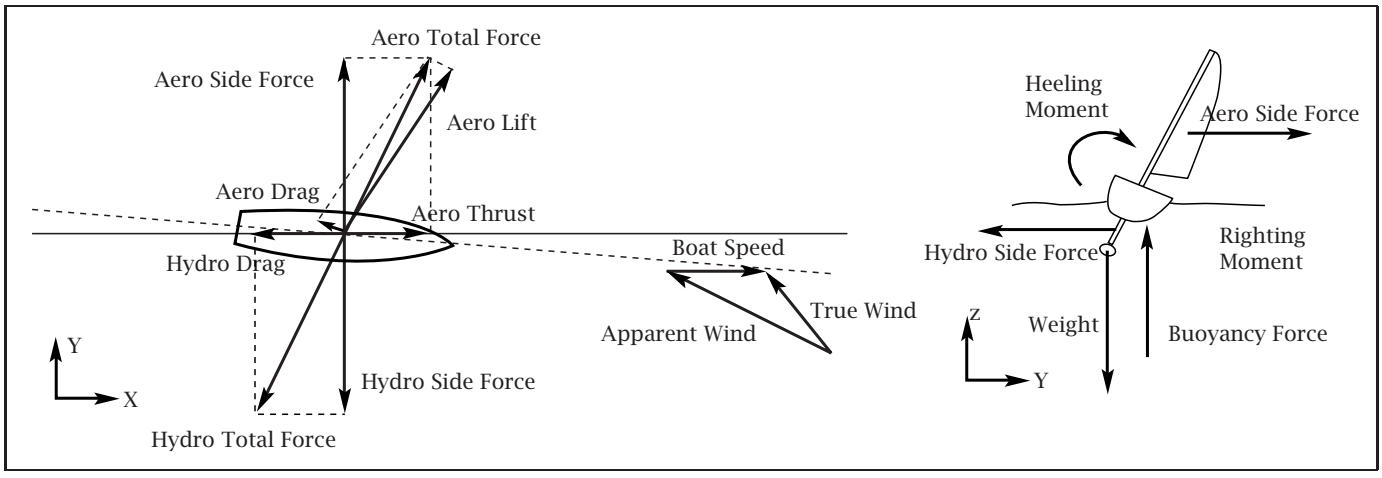

Figure 10. Forces and moments acting on boat.

The role of advanced computational fluid dynamics is to supply accurate estimates of the forces acting on the boat in different sailing conditions in order to improve the reliability of the prediction of the overall performance associated with a given design configuration.

The flow equations

Let Ω denote the three-dimensional computational domain in which we solve the flow equations. If $\hat{\Omega}$ is a region surrounding the boat B, the computational domain is the complement of B with respect to $\hat{\Omega}$, that is $\Omega = \hat{\Omega} \backslash B$. The equations that govern the flow around B are the density-dependent (or inhomogeneous) incompressible Navier–Stokes equations, which read:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0 \tag{2}$$

$$\frac{\partial (\rho u)}{\partial t} + \nabla \cdot (\rho u \otimes u) - \nabla \cdot \tau(u,p) = \rho g \tag{3}$$

$$\nabla \cdot u = 0 \tag{4}$$

for $x \in \Omega$ and $0 < t < T$, and where ρ is the (variable) density, u is the velocity field, p is the pressure, $g = (0,0,g)^T$ is the gravity acceleration, and $\tau(u,p) = \mu(\nabla u + \nabla u^T) - pI$ is the stress tensor with μ indicating the (variable) viscosity. The above equations have to be complemented with suitable initial conditions and boundary conditions. For the latter we typically consider a given velocity profile at the inflow boundary, with a flat far field free-surface elevation.

In the case we are interested in, the computational domain Ω is made of two regions, the volume Ω_w occupied by the water and that Ω_a occupied by the air. The interface Γ separating Ω_w from Ω_a is the (unknown) free-surface, which may be a disconnected two-dimensional manifold if wave breaking is accounted for. The unknown density ρ actually takes two constant states, ρ_w (in Ω_w) and ρ_a (in Ω_a). The values of ρ_w and ρ_a depend on the fluid temperatures, which are considered to be constant in the present model. The fluid viscosities μ_w (in Ω_w) and μ_a (in Ω_a) are constants that depend on ρ_w and ρ_a, respectively.

The set of equations (2)-(4) can therefore be seen as a model for the evolution of a two-phase flow consisting of two immiscible incompressible fluids with constant densities ρ_w and ρ_a and different viscosity coefficients μ_w and μ_a. In this respect, in view of the numerical simulation, we could regard equation (2) as the candidate for updating the (unknown) interface location Γ, then treat equations (3)-(4) as a coupled system of Navier–Stokes equations in the two sub-domains Ω_w and Ω_a:

$$\frac{\partial (\rho_w u_w)}{\partial t} + \nabla \cdot (\rho_w u_w \otimes u_w) - \nabla \cdot \tau_w(u_w, p_w) = \rho_w g,$$

$$\nabla \cdot u_w = 0,$$

in $\Omega_w \times (0,T)$,

$$\frac{\partial (\rho_a u_a)}{\partial t} + \nabla \cdot (\rho_a u_a \otimes u_a) - \nabla \cdot \tau_a(u_a, p_a) = \rho_a g,$$

$$\nabla \cdot u_a = 0,$$

in $\Omega_a \times (0,T)$. We have set $\tau_w(u_w,p_w) = \mu_w(\nabla u_w + \nabla u_w{}^T) - p_w I$, while $\tau_a(u_a,p_a)$ is defined similarly.

The free surface Γ is a sharp interface between Ω_w and Ω_a, on which the normal components of the two velocities $u_a \cdot n$ and $u_w \cdot n$ should agree. Furthermore, the tangential components must match as well since the two flows are incompressible. Thus we have the following kinematic condition

$$u_a = u_w \quad \text{on } \Gamma. \tag{5}$$

Moreover, the forces acting on the fluid at the free-surface are in equilibrium. This is a dynamic

condition and means that the normal forces on either side of Γ are of equal magnitude and opposed direction, while the tangential forces must agree in both magnitude and direction:

$$\tau_a(u_a, p_a) \cdot n = \tau_w(u_w, p_w) \cdot n + \kappa\sigma n \quad \text{on } \Gamma, \tag{6}$$

where σ is the surface tension coefficient, that is a force per unit length of a free surface element acting tangentially to the free-surface. It is a property of the liquid and depends on the temperature as well as on other factors. The quantity κ in (6) is the curvature of the free-surface, $\kappa = R_{t_1}^{-1} + R_{t_2}^{-1}$, where R_{t_1} and R_{t_2} are radii of curvature along the coordinates (t_1, t_2) of the plane tangential to the free-surface (orthogonal to n).

Coupling with a 6-DOF rigid body dynamical system

The attitude of the boat advancing in calm water or wavy sea is strictly correlated with its performance. For this reason, a state-of-the-art numerical tool for yacht design predictions should be able to account for the boat motion.

Following the approach adopted in [2, 3], two orthogonal cartesian reference systems are considered: an inertial reference system (O, X, Y, Z), which moves forward with the mean boat speed, and a body-fixed reference system (G, x, y, z), whose origin is the boat center of mass G, which translates and rotates with the boat. The XY plane in the inertial reference system is parallel to the undisturbed water surface, and the Z-axis points upward. The body-fixed x-axis is directed from bow to stern, y positive starboard, and z upwards.

The dynamics of the boat in the 6 degrees of freedom are determined by integrating the equations of variation of linear and angular momentum in the inertial reference system, as follows

$$m\ddot{X}_G = F \tag{7}$$

$$\bar{\bar{T}}\bar{\bar{I}}\bar{\bar{T}}^{-1}\dot{\Omega} + \Omega \times \bar{\bar{T}}\bar{\bar{I}}\bar{\bar{T}}^{-1}\Omega = M_G \tag{8}$$

where m is the boat mass, $\ddot{X}_G$ is the linear acceleration of the center of mass, F is the force acting on the boat, $\dot{\Omega}$ and Ω are the angular acceleration and velocity, respectively, M_G is the moment with respect to G acting on the boat, $\bar{\bar{I}}$ is the tensor of inertia of the boat about the body-fixed reference system axes, and $\bar{\bar{T}}$ is the transformation matrix between the body-fixed and the inertial reference system (see [2] for details).

The forces and moments acting on the boat are given by

$$\begin{aligned} F &= F_{\text{Flow}} + mg + F_{\text{Ext}} \\ M_G &= M_{\text{Flow}} + (X_{\text{Ext}} - X_G) \times F_{\text{Ext}} \end{aligned}$$

where F_{Flow} and M_{Flow} are the force and moment, respectively, due to the interaction with the flow and F_{Ext} is an external forcing term (which may model, e.g., the wind force on sails) while X_{Ext} is its application point.

The equations for wind–sails interaction

The sail deformation is due to the fluid motion: the aerodynamic pressure field deforms the sail surfaces and this, in its turn, modifies the flow field around the sails.

From a mathematical viewpoint, this yields a coupled system that comprises the incompressible Navier-Stokes equations with constant density $\rho = \rho_{air}$ (3-4) and a second order elastodynamic equation that models the sail deformation as that of a membrane. More specifically, the evolution of the considered elastic structure is governed by the classical conservation laws for continuum mechanics.

Considering a Lagrangian framework, if $\hat{\Omega}_s$ is the reference 2D domain occupied by the sails, the governing equation can be written as follows:

$$\rho_s \frac{\partial^2 d}{\partial t^2} = \nabla \cdot \sigma_s(d) + f_s \quad in \quad \hat{\Omega}_s \times (0, T], \tag{9}$$

where ρ_s is the material density, the displacement d is a function of the space coordinates $x \in \hat{\Omega}_s$ and of the time $t \in [0; T]$, σ_s are the internal stresses while f_s are the external loads acting on the sails (these are indeed the normal stresses $\tau(u, p) \cdot n$ on the sail surface exerted by the flowfield). In fact, $\hat{\Omega}_s$ represents a wider (bounded and disconnected) domain that includes also the mast and the yarns as parts of the structural model. The boundary of $\hat{\Omega}_s$ is denoted by $\partial\hat{\Omega}_s$ and $[0; T] \subset \mathbb{R}^+$ is the time interval of our analysis. For suitable initial and boundary conditions and an assignment of an appropriate constitutive equation for the considered materials (defining $\sigma_s(d)$), the displacement field d is computed by solving (9) in its weak form:

$$\begin{aligned} \int_{\hat{\Omega}_s} \rho_s \frac{\partial^2 d_i}{\partial t^2} (\delta d_i) dx + \int_{\hat{\Omega}_s} \sigma^{II}{}_{ik} (\delta\epsilon_{ki}) dx \\ = \int_{\hat{\Omega}_s} f_{s\,i} (\delta d_i) dx, \end{aligned} \tag{10}$$

where σ^{II} is the second Piola-Kirchoff stress tensor, ϵ is the Green-Lagrange strain tensor, and δd are the test functions expressing the virtual deformation. The second coupling condition enforces the continuity of the two velocity fields, u and $\frac{\partial d}{\partial t}$, on the sail surface.

Fluid-structural coupling algorithm

As previously introduced, the coupling procedure iteratively loops between the fluid solver (passing sail velocities and getting pressure fields) and the structural solver (passing pressures and getting velocities and structural deformations) until the structure undergoes no more deformations because a perfect balance of forces and moments is reached. When dealing with transient simulations,

this must be true for each time step, and the sail geometry evolves over time as a sequence of converged states. On the other hand, a steady simulation can be thought of as a transient one with an infinite time step, such that "steady" means a sort of average of the true (unsteady) solution over time. More formally, we can define two operators called `Fluid` and `Struct` that represent the fluid and structural solvers, respectively. In particular, `Fluid` can be any procedure that can solve the incompressible Navier-Stokes equations while `Struct` should solve a membrane-like problem, possibly embedding suitable nonlinear models to take into account complex phenomena such as, for example, the structural reactions due to a fabric wrinkle.

The fixed-point problem can be reformulated with the new operators as follows:

$$\texttt{Fluid}(\texttt{Struct}(p)) = p. \tag{11}$$

A resolving algorithm can be devised as follows. At a given iteration the pressure field on sails p is passed to the structural solver (`Struct`), which returns the new sail geometries and the new sail velocity fields. Afterwards, these quantities are passed to the fluid solver (`Fluid`) which returns the same pressure field p on sails. Clearly, the "equal" sign holds only at convergence. The resulting fixed-point iteration can be rewritten more explicitly as follows: Given a pressure field on sails p_k, do:

$$\begin{aligned} (G_{k+1}, \mathbf{U}_{k+1}) &= \texttt{Struct}(p_k) \\ \bar{p}_{k+1} &= \texttt{Fluid}(G_{k+1}, \mathbf{U}_{k+1}) \\ p_{k+1} &= (1-\theta_k)p_k + \theta_k \bar{p}_{k+1} \end{aligned} \tag{12}$$

where G_{k+1} and $\mathbf{U}_{k+1}$ are the sail geometry and the sail velocity field at step $k+1$, respectively, while θ_k is a suitable acceleration parameter.

Even though the final goal is to run an unsteady simulation, the fluid-structure procedure has to run some preliminary steady couplings to provide a suitable initial condition. The steady run iterates until a converged sail shape and flow field are obtained, where converged means that there does exist a value of k_c such that (11) is satisfied for every $k > k_c$ (within given tolerances on forces and/or displacements). When running steady simulations the velocity of the sails is required to be null at each coupling, thus somehow enforcing the convergence condition (which prescribes null velocities at convergence). This explains why convergence is slightly faster when running steady simulations with respect to transient ones (clearly only when such a solution reflects a steady state physical solution).

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